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Solutions to exercise set 1

1 First note that the function has period 2π. The Fourier coefficients a_n and b_n of a function with period 2π is given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \tag{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \tag{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x. \tag{3}$$

Using the orthogonality property of trigonometric system [1, Theorem 1 p. 479], some work can be spared by extracting these coefficients directly from the function:

$$a_0 = 5$$

$$a_n = \begin{cases} -4 & n = 2 \\ 5 & n = 8 \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \begin{cases} -2 & n = 5 \\ 0 & \text{otherwise.} \end{cases}$$

2 **a)** A function has period p if $f(x+p) = f(x)$ for all x. Consider first the function $g(x) = f(kx)$ and define $\alpha = p/k$. Then,

$$g(x+\alpha) = f(k(x+\alpha)) = f(k(x+p/k)) = f(kx+p) = f(kx) = g(x)$$

where the the periodicity of f has been used. Thus, $f(kx)$ has period p/k. Similarly for the function $h(x) = f(x/k)$ and $\beta = kp$

$$h(x+\beta) = f((x+\beta)/k) = f((x+pk)/k) = f(x/k+p) = f(x/k) = h(x).$$

Thus, $f(x/k)$ has period pk.

b) By induction we have

$$\begin{aligned} f(x+kp) &= f(x+p+(k-1)p) = f(x+(k-1)p) = f(x+(k-2)p) \\ &= \cdots = f(x+p) = f(x) \end{aligned}$$

Consider $g(x) = f(kx)$. Then,

$$g(x+p) = f(k(x+p)) = f(kx+kp) = f(kx) = g(x).$$

Thus, $f(kx)$ has also period p.

c) From the previous exercise $g(x) = f(3x)$ has also period 2π and can be written as

$$g(x) = \tilde{a}_0 + \sum_{n=1}^{\infty} \tilde{a}_n \cos nx + \tilde{b}_n \sin nx$$

The goal would be to find the relation between the coefficients $\{a_n, b_n\}$ and $\{\tilde{a}_n, \tilde{b}_n\}$. From the equality $g(x) = f(3x)$ we get

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 3nx + b_n \sin 3nx.$$

Using the same trick as in task 1, we can simply extract the coefficients from this expression

$$\begin{aligned}
\tilde{a}_0 &= a_0 \\
\tilde{a}_n &= \begin{cases} a_{n/3} & \text{whenever } n \text{ is divisible by 3} \\ 0 & \text{otherwise} \end{cases} \\
\tilde{b}_n &= \begin{cases} b_{n/3} & \text{whenever } n \text{ is divisible by 3} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$\boxed{3}$ **a)** First note that $f(x)$ is an even function, which implies that $b_n = 0$ for all n in (3). The coefficient a_0 in (1) is given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} \sin(x)\,\mathrm{d}x = \frac{1}{\pi} [-\cos x]_0^{\pi} = \frac{2}{\pi}.$$

Finally, using the trigonometric identity

$$\sin[(n+1)x] - \sin[(n-1)x] = 2 \sin x \cos nx$$

we get

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,\mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\,\mathrm{d}x = \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos nx\,\mathrm{d}x \\
&= \frac{1}{\pi} \int_{0}^{\pi} \sin[(n+1)x] - \sin[(n-1)x]\,\mathrm{d}x \\
&= \frac{1}{\pi} \left[-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1} \right]_0^{\pi} \\
&= \frac{1}{\pi} \left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right) = -\frac{2}{\pi} \frac{(-1)^n + 1}{n^2 - 1} \\
&= \begin{cases} -\frac{4}{\pi} \frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}
\end{aligned}$$

where we have used $\cos n\pi = (-1)^n$. The Fourier series of $f(x)$ is thus given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx.$$

b) By evaluating the Fourier series above wisely at $x = 0$ we get

$$f(0) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

Since we also have $f(0) = 0$, we find

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

$\boxed{4}$ **a)** This function is neither odd nor even, and so all coefficients in (1), (2) and (3). We first compute a_0

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2\pi} \int_0^{\pi} x(\pi - x)\,\mathrm{d}x = \frac{1}{2\pi} \left[\frac{\pi}{2} x^2 - \frac{1}{3} x^3 \right]_0^{\pi} = \frac{\pi^2}{12}.$$

The following integrals is needed to compute a_n and b_n

$$\int_0^{\pi} x \cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n} x \sin nx \right]_0^{\pi}}_{=0} - \frac{1}{n} \int_0^{\pi} \sin nx\,\mathrm{d}x = \left[\frac{1}{n^2} \cos nx \right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} x \sin nx\,\mathrm{d}x = \left[-\frac{1}{n} x \cos nx \right]_0^{\pi} + \frac{1}{n} \underbrace{\int_0^{\pi} \cos nx\,\mathrm{d}x}_{=0} = -\frac{\pi}{n}(-1)^n$$

$$\int_0^{\pi} x^2 \cos nx\,\mathrm{d}x = \underbrace{\left[\frac{1}{n} x^2 \sin nx \right]_0^{\pi}}_{=0} - \frac{2}{n} \underbrace{\int_0^{\pi} x \sin nx\,\mathrm{d}x}_{=-\frac{\pi}{n}(-1)^n} = \frac{2\pi}{n^2}(-1)^n$$

$$\int_0^{\pi} x^2 \sin nx\,\mathrm{d}x = \left[-\frac{1}{n} x^2 \cos nx \right]_0^{\pi} + \frac{2}{n} \underbrace{\int_0^{\pi} x \cos nx\,\mathrm{d}x}_{=\frac{(-1)^n - 1}{n^2}}$$
$$= -\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1 \right].$$

We now have

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,\mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} x(\pi - x) \cos nx\,\mathrm{d}x \\
&= \int_0^{\pi} x \cos nx\,\mathrm{d}x - \frac{1}{\pi} \int_0^{\pi} x^2 \cos nx\,\mathrm{d}x = \frac{(-1)^n - 1}{n^2} - \frac{2}{n^2}(-1)^n \\
&= \begin{cases} -\frac{2}{n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,\mathrm{d}x = \frac{1}{\pi} \int_0^{\pi} x(\pi - x) \sin nx\,\mathrm{d}x \\
&= \int_0^{\pi} x \sin nx\,\mathrm{d}x - \frac{1}{\pi} \int_0^{\pi} x^2 \sin nx\,\mathrm{d}x \\
&= -\frac{\pi}{n}(-1)^n - \frac{1}{\pi} \left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}\left[(-1)^n - 1 \right] \right] = \begin{cases} \frac{4}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}
\end{aligned}$$

The Fourier series of $f(x)$ is thus

$$f(x) = \frac{\pi^2}{12} - \sum_{n=1}^{\infty} \frac{1}{2n^2} \cos 2nx + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin[(2n-1)x]$$

b) By evaluating the Fourier series above wisely at $x = 0$ we get

$$f(0) = \frac{\pi^2}{12} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since we also have $f(0) = 0$, we find

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

References

[1] E. Kreyszig, *Advanced engineering mathematics*, 10th edition, John Wiley & Sons, 2011.