Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4130
Matematikk 4N
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**Solutions to exercise set 7**

**1** **a)** Using the definition of Laplace transform [1, (1) p. 204], we have (using the substitution $\tau = t - np$ and the periodicity of $f$, $f(\tau + np) = f(\tau)$)

$$\begin{aligned}
\mathscr{L}\{f\} &= \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \sum_{n=0}^\infty \int_{np}^{np+p} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \sum_{n=0}^\infty \int_0^p f(\tau+np)\mathrm{e}^{-s(\tau+np)}\,\mathrm{d}\tau \\
&= \sum_{n=0}^\infty \mathrm{e}^{-snp}\int_0^p f(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau = \int_0^p f(t)\mathrm{e}^{-st}\,\mathrm{d}t \sum_{n=0}^\infty \left(\mathrm{e}^{-sp}\right)^n \\
&= \frac{1}{1-\mathrm{e}^{-ps}}\int_0^p f(t)\mathrm{e}^{-st}\,\mathrm{d}t
\end{aligned}$$

where we have used the following summation formula for geometric series was applied in the final equality

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x}, \qquad |x| < 1.$$

The condition $|\mathrm{e}^{-ps}| < 1$ is satisfied since $s, p > 0$.

**b)** With $f(t) = t$ we get (using partial integration)

$$\begin{aligned}
\mathscr{L}\{f\} &= \frac{1}{1-\mathrm{e}^{-ps}}\int_0^p f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{1-\mathrm{e}^{-ps}}\int_0^p t\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \frac{1}{1-\mathrm{e}^{-ps}}\left(\left[-\frac{t}{s}\mathrm{e}^{-st}\right]_0^p + \frac{1}{s}\int_0^p \mathrm{e}^{-st}\,\mathrm{d}t\right) \\
&= \frac{1}{1-\mathrm{e}^{-ps}}\left(-\frac{p}{s}\mathrm{e}^{-sp} - \frac{1}{s^2}\left[\mathrm{e}^{-st}\right]_0^p\right) = \frac{1}{1-\mathrm{e}^{-ps}}\left(\frac{1-\mathrm{e}^{-ps}}{s^2} - \frac{p}{s}\mathrm{e}^{-ps}\right) \\
&= \frac{1}{s^2} - \frac{p}{1-\mathrm{e}^{-ps}}\frac{\mathrm{e}^{-ps}}{s}
\end{aligned}$$

**2** The RLC-circuit is govern by the following integro-differential equation [1, (1') p. 94]

$$Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau = v(t) \tag{1}$$

where (in our case)

$$\begin{aligned}
v(t) &= \begin{cases} -34\mathrm{e}^{-t} & 0 < t < 4 \\ 0 & \text{otherwise} \end{cases} \\
&= 34\mathrm{e}^{-t}(1 - u(t-4)) = 34\mathrm{e}^{-t} - 34\mathrm{e}^{-4}\mathrm{e}^{-(t-4)}u(t-4)).
\end{aligned}$$

The Laplace transform of this expression is given by (using $t$-shifting [1, (4) p. 219] and [1, (7) p. 249])

$$V(s) = \mathscr{L}\{v(t)\} = \frac{34}{s+1} - \frac{34\mathrm{e}^{-4}\mathrm{e}^{4s}}{s+1} = \frac{34\left(1-\mathrm{e}^{-4s-4}\right)}{s+1}.$$

Equating this result with the Laplace transform of the left hand side of (1), we obtain (using [1, (1) p. 211] and [1, (4) p. 213])

$$\begin{aligned}\frac{34\left(1-\mathrm{e}^{-4s-4}\right)}{s+1} &= \mathscr{L}\left\{Li'(t) + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,\mathrm{d}\tau\right\} \\ &= L\left(sI(s) - i(0)\right) + RI(s) + \frac{1}{C}\frac{I(s)}{s} \\ &= sI(s) + 4I(s) + 20\frac{I(s)}{s} = \left(s + 4 + \frac{20}{s}\right) I(s).\end{aligned}$$

Thus,

$$I(s) = \frac{34s\left(1-\mathrm{e}^{-4s-4}\right)}{(s+1)(s^2+4s+20)}.$$

Using partial fraction expansion we get

$$\begin{aligned}\frac{34s}{(s+1)(s^2+4s+20)} &= \frac{A}{s+1} + \frac{B+Ds}{s^2+4s+20} \\ &= \frac{(A+D)s^2 + (4A+B+D)s + 20A + B}{(s+1)(s^2+4s+20)} \\ &\Rightarrow \quad \begin{cases} A + D = 0 \\ 4A + B + D = 34 \\ 20A + B = 0 \end{cases} \\ &\Rightarrow \quad A = -2, \quad B = 40, \quad D = 2.\end{aligned}$$

Hence,

$$\begin{aligned}I(s) &= \left(-\frac{2}{s+1} + \frac{40+2s}{s^2+4s+20}\right)\left(1-\mathrm{e}^{-4s-4}\right) \\ &= \left(-\frac{2}{s-(-1)} + \frac{2(s-(-2))}{(s-(-2))^2+4^2} + 9\cdot\frac{4}{(s-(-2))^2+4^2}\right)\left(1-\mathrm{e}^{-4}\mathrm{e}^{-4s}\right).\end{aligned}$$

The inverse Laplace transform of this functions yields the final result (using $s$-shifting [1, Theorem 2 p. 208], $t$-shifting [1, Theorem 1 p. 219] and [1, (7) and (13) and (14) p. 249])

$$\begin{aligned}i(t) &= -2\mathrm{e}^{-1\cdot t} + 2\mathrm{e}^{-2t}\cos 4t + 9\mathrm{e}^{-2t}\sin 4t \\ &\quad - u(t-4)\mathrm{e}^{-4}\left(-2\mathrm{e}^{-1\cdot(t-4)} + 2\mathrm{e}^{-2(t-4)}\cos[4(t-4)] + 9\mathrm{e}^{-2(t-4)}\sin[4(t-4)]\right) \\ &= -2\mathrm{e}^{-t} + \mathrm{e}^{-2t}\left(2\cos 4t + 9\sin 4t\right) \\ &\quad - u(t-4)\left(-2\mathrm{e}^{-t} + \mathrm{e}^{-2(t-2)}\left(2\cos[4(t-4)] + 9\sin[4(t-4)]\right)\right).\end{aligned}$$

$\boxed{3}$ **a)** Taking the Laplace transform of

$$y''(t) + 4y'(t) + 5y(t) = \delta(t-1)$$

yields (using [1, Theorem 1 p. 211] and [1, (35) p. 250])

$$s^2Y(s) - sy(0) - y'(0) + 4\,(sY(s) - y(0)) + 5Y(s) = \mathrm{e}^{-s}.$$

With the initial conditions $y(0) = 0$ and $y'(0) = 3$ we have

$$s^2Y(s) - 3 + 4sY(s) + 5Y(s) = \mathrm{e}^{-s}$$

such that

$$Y(s) = \frac{\mathrm{e}^{-s} + 3}{s^2 + 4s + 5} = \frac{3}{(s - (-2))^2 + 1^2} + \frac{\mathrm{e}^{-1\cdot s}}{(s - (-2))^2 + 1^2}$$

Taking the inverse Laplace transform (using $s$-shifting [1, Theorem 2 p. 208], $t$-shifting [1, Theorem 1 p. 219] and [1, (13) p. 249])

$$y(t) = 3\mathrm{e}^{-2t}\sin t + \mathrm{e}^{-2(t-1)}\sin(t-1)u(t-1)$$

**b)** Taking the Laplace transform of

$$\begin{aligned} y''(t) + 5y'(t) + 6y(t) &= \delta\left(t - \frac{\pi}{2}\right) + u(t-\pi)\cos t \\ &= \delta\left(t - \frac{\pi}{2}\right) - u(t-\pi)\cos(t-\pi) \end{aligned}$$

yields (using [1, Theorem 1 p. 211], [1, Theorem 1 p. 219], [1, (14) p. 249] and [1, (34) and (35) p. 250])

$$s^2Y(s) - sy(0) - y'(0) + 5\,(sY(s) - y(0)) + 6Y(s) = \mathrm{e}^{-\pi s/2} - \frac{s\mathrm{e}^{-\pi s}}{s^2+1}.$$

With the initial conditions $y(0) = 0$ and $y'(0) = 0$ we have

$$s^2Y(s) + 5sY(s) + 6Y(s) = \mathrm{e}^{-\pi s/2} - \frac{s\mathrm{e}^{-\pi s}}{s^2+1}$$

such that

$$Y(s) = \frac{\mathrm{e}^{-\pi s/2}}{s^2+5s+6} - \frac{s\mathrm{e}^{-\pi s}}{(s^2+1)(s^2+5s+6)}.$$

Using partial fraction expansion we get $(s^2 + 5s + 6 = (s+2)(s+3))$

$$\begin{aligned} \frac{1}{s^2+5s+6} &= \frac{A}{s+2} + \frac{B}{s+3} = \frac{(A+B)s + 3A + 2B}{s^2+5s+6} \\ &\Rightarrow \begin{cases} A + B = 0 \\ 3A + 2B = 1 \end{cases} \\ &\Rightarrow \quad A = 1, \quad B = -1 \end{aligned}$$

and

$$\begin{aligned} &\frac{s}{(s^2+1)(s^2+5s+6)} = \frac{A+Bs}{s^2+1} + \frac{C}{s+2} + \frac{D}{s+3} = \frac{A+Bs}{s^2+1} + \frac{(C+D)s + 3C + 2D}{(s^2+5s+6)} \\ &= \frac{(B+C+D)s^3 + (A+5B+3C+2D)s^2 + (5A+6B+C+D)s + 6A + 3C + 2D}{(s^2+1)(s^2+5s+6)} \\ &\Rightarrow \begin{cases} B + C + D = 0 \\ A + 5B + 3C + 2D = 0 \\ 5A + 6B + C + D = 1 \\ 6A + 3C + 2D = 0 \end{cases} \\ &\Rightarrow \quad A = \frac{1}{10}, \quad B = \frac{1}{10}, \quad C = -\frac{2}{5}, \quad D = \frac{3}{10}. \end{aligned}$$

Hence,

$$
\begin{aligned}
Y(s) &= \left(\frac{1}{s+2} - \frac{1}{s+3}\right) \mathrm{e}^{-\pi s/2} - \left(\frac{1}{10}\frac{1+s}{s^2+1} - \frac{2}{5}\frac{1}{s+2} + \frac{3}{10}\frac{1}{s+3}\right) \mathrm{e}^{-\pi s} \\
&= \left(\frac{1}{s+2} - \frac{1}{s+3}\right) \mathrm{e}^{-s\pi/2} - \frac{1}{10}\left(\frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{4}{s+2} + \frac{3}{s+3}\right) \mathrm{e}^{-\pi s}
\end{aligned}
$$

Taking the inverse Laplace transform (using $t$-shifting [1, Theorem 1 p. 219] and [1, (7) and (13) and (14) p. 249])

$$
\begin{aligned}
y(t) &= \left(\mathrm{e}^{-2(t-\pi/2)} - \mathrm{e}^{-3(t-\pi/2)}\right) u\left(t - \frac{\pi}{2}\right) \\
&\quad - \frac{1}{10}\left(\sin(t-\pi) + \cos(t-\pi) - 4\mathrm{e}^{-2(t-\pi)} + 3\mathrm{e}^{-3(t-\pi)}\right) u(t-\pi) \\
&= \left(\mathrm{e}^{-2(t-\pi/2)} - \mathrm{e}^{-3(t-\pi/2)}\right) u\left(t - \frac{\pi}{2}\right) \\
&\quad + \frac{1}{10}\left(\sin t + \cos t + 4\mathrm{e}^{-2(t-\pi)} - 3\mathrm{e}^{-3(t-\pi)}\right) u(t-\pi)
\end{aligned}
$$

## References

[1] E. Kreyszig, *Advanced engineering mathematics*, 10th edition, John Wiley & Sons, 2011.