Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let M be the plane of $\mathbb{R}^3$ given by $x_1 + x_2 + x_3 = 0$. Find the linear mapping that is the orthogonal projection of $\mathbb{R}^3$ onto this plane.

Solution. We defined the projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ by decomposing each $x \in \mathbb{R}^3$ into $x = y + z$ using the projection theorem, where $y \in M$ and $z \in M^\perp$, and defining P by $Px = y$. A natural starting point is therefore to find $M^\perp$. Since $\mathbb{R}^3$ is 3-dimensional, M is 2-dimensional and $\mathbb{R}^3 = M \oplus M^\perp$ by the projection theorem, $M^\perp$ must be one-dimensional. It is not difficult to see that the vector $a = (1, 1, 1)$ belongs to $M^\perp$, since[1]

$$\langle x, a\rangle = x_1 + x_2 + x_3 = 0 \text{ if } x \in M.$$

Since $M^\perp$ is one-dimensional and contains a, it follows that $M^\perp = \{\lambda a : \lambda \in \mathbb{R}\}$. Now let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We want to write decompose $x = y + z$ with $y \in M$ and $z \in M^\perp$. Since $M^\perp = \{\lambda a : \lambda \in \mathbb{R}\}$, we must have that $z = \lambda a$ for some $\lambda \in \mathbb{C}$:

$$x = y + \lambda a,$$

so in terms of coordinates we have

$$(x_1, x_2, x_3) = (y_1, y_2, y_3) + (\lambda, \lambda, \lambda).$$

We are looking for (y_1, y_2, y_3), and need to eliminate λ. To achieve this we have one unused assumption: $y \in M$, so $y_1 + y_2 + y_3 = 0$. To use this, we solve for (y_1, y_2, y_3):

$$(y_1, y_2, y_3) = (x_1 - \lambda, x_2 - \lambda, x_3 - \lambda).$$

Since $(y_1, y_2, y_3) \in M$ we must have that $0 = y_1 + y_2 + y_3 = x_1 - \lambda + x_2 - \lambda + x_3 - \lambda = x_1 + x_2 + x_3 - 3\lambda$. We may solve this for λ to get

$$\lambda = \frac{x_1 + x_2 + x_3}{3},$$

[1] Remember that the inner product on $\mathbb{R}^3$ is just the usual dot product.

and inserting this back into our expression for (y_1, y_2, y_3) we find that

$$\begin{aligned}(y_1, y_2, y_3) &= (x_1 - \lambda, x_2 - \lambda, x_3 - \lambda) \\ &= \frac{1}{3}(2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2).\end{aligned}$$

Since $Px = y$, this means that we have shown that

$$P(x_1, x_2, x_3) = \frac{1}{3}(2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2).$$

2 Let $A \subset \mathbb{R}$ be a set.

Prove that if A is bounded from below, then there is a sequence $(a_n) \subset A$ such that $a_n \to \inf A$ as $n \to \infty$. (In other words, prove that $\inf A \in \bar{A}$.)

Similarly, if A is bounded from above, prove that there is a sequence $(a_n) \subset A$ such that $a_n \to \sup A$ as $n \to \infty$. (In other words, prove that $\sup A \in \bar{A}$.)

Solution. Let $A \subset \mathbb{R}$ be bounded from below. First off, because of the least upper bound property, the infimum of A exists (as a real number), that is, $\inf A \in \mathbb{R}$. [2]

Let $\epsilon > 0$. Since $\inf A$ is the *greatest* lower bound of A, we have that $\inf A + \epsilon$ cannot be a lower bound for A, so there is some element $a_\epsilon \in A$ such that $a_\epsilon < \inf A + \epsilon$. Furthermore, since $\inf A$ is a lower bound of A, and since $a_\epsilon \in A$, we must have that $\inf A \leq a_\epsilon$. Thus

$$\inf A \leq a_\epsilon < \inf A + \epsilon\,.$$

For every $n \geq 1$ we may choose $\epsilon = \frac{1}{n}$. Using the above, there is some element $a_n \in A$ such that

$$\inf A \leq a_n < \inf A + \frac{1}{n}\,.$$

We have obtained a sequence $(a_n) \subset A$ such that $\inf A \leq a_n < \inf A + \frac{1}{n}$ for all $n \geq 1$. Subtracting $\inf A$ from all sides of this inequality, we get that

$$0 \leq a_n - \inf A < \frac{1}{n}\,,$$

which implies

$$|a_n - \inf A| \leq \frac{1}{n} \to 0 \quad \text{as } \ n \to \infty, \qquad \text{so } \ a_n \to \inf A\,.$$

[2] I'm afraid that we did not show that the least upper bound property implies the existence of the infinum, and I would not require this to be part of the student's solution. The student should, however, be made aware that the existence of the infinum of a set bounded from below follows from the least upper bound property.

The second part, regarding $\sup A$ is very similar. Let $A \subset \mathbb{R}$ be bounded from above. By the least upper bound property, the supremum of A exists (as a real number), that is, $\sup A \in \mathbb{R}$.

Let $\epsilon > 0$. Since $\sup A$ is the *least* upper bound of A, we have that $\sup A - \epsilon$ cannot be an upper bound for A, so there is some element $a_\epsilon \in A$ such that $a_\epsilon > \sup A - \epsilon$. Furthermore, since $\sup A$ is an upper bound of A, and since $a_\epsilon \in A$, we must have that $a_\epsilon \leq \sup A$. Thus

$$\sup A - \epsilon < a_\epsilon \leq \sup A\,.$$

For every $n \geq 1$ we may choose $\epsilon = \frac{1}{n}$. Using the above, there is some element $a_n \in A$ such that

$$\sup A - \frac{1}{n} < a_n \leq \sup A\,.$$

We have obtained a sequence $(a_n) \subset A$ such that $\sup A - \frac{1}{n} < a_n \leq \sup A$ for all $n \geq 1$. Subtracting $\sup A$ from all sides of this inequality, we get that

$$-\frac{1}{n} < a_n - \sup A \leq 0\,,$$

which implies

$$|a_n - \sup A| \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty, \quad \text{so } a_n \to \sup A\,.$$

3 Let T be a bounded linear operator on a Hilbert spae X. Show that the operator norm of T can be expressed in terms of the innerproduct of X:

$$\|T\| = \sup\{\langle Tx, y\rangle : x, y \in X \;\text{ with }\; \|x\| = \|y\| = 1\}.$$

Solution. We will first show that

$$\sup\{\langle x, y\rangle : y \in X \;\text{ with }\; \|y\| = 1\} = \|x\| \;\text{ for all }\; x \in X. \tag{1}$$

By the Cauchy-Schwarz inequality we have

$$\langle x, y\rangle \leq |\langle x, y\rangle| \leq \|x\|\|y\| = \|x\|, \;\text{ for all }\; x, y \in X \text{ with } \|y\| = 1.$$

It follows that

$$\sup\{\langle x, y\rangle : y \in X \;\text{ with }\; \|y\| = 1\} \leq \|x\| \;\text{ for all }\; x \in X.$$

It remains to show the inequality

$$\sup\{\langle x, y\rangle : y \in X \;\text{ with }\; \|y\| = 1\} \geq \|x\| \;\text{ for all }\; x \in X.$$

This clearly holds when $x = 0$, since $\langle 0, y\rangle = 0$ for all $y \in X$. Now suppose $x \neq 0$. Let $y = \frac{x}{\|x\|}$ and notice that $\|y\| = 1$. We have

$$\langle x, y\rangle = \langle x, \frac{x}{\|x\|}\rangle = \frac{1}{\|x\|}\langle x, x\rangle = \frac{1}{\|x\|}\|x\|^2 = \|x\|.$$

The inequality follows.

We will now show that the norm of T can be expressed in terms of the innerproduct. We have

$$\begin{aligned}\|T\| &= \sup\{\|Tx\|,\, x \in X \ \text{ with } \ \|x\| = 1\}\\ &= \sup\{\,\sup\{\langle Tx, y\rangle : \, y \in X,\, \|y\| = 1\} : \, x \in X,\, \|x\| = 1\} \quad \text{(equation (1))}\\ &= \sup\{\langle Tx, y\rangle : \, x, y \in X \ \text{ with } \ \|x\| = \|y\| = 1\}\end{aligned}$$

4 Let $M = \{x \in \ell^2 : x = (x_1, 0, x_3, 0, x_5, ...)\}$ be the subspace of odd sequences in ℓ^2. Determine the orthogonal complement $M^\perp$. You must prove that the space you find really is the orthogonal complement of M.

Solution. We claim that $M^\perp = M_e$, where $M_e = \{x \in \ell^2 : x = (0, x_2, 0, x_4, 0, x_6...)\}$ is the set of even sequences in ℓ^2.
First assume that $x \in M_e$. If $y = (y_1, 0, y_3, 0, y_5, ...) \in M$, then

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i = \sum_{i \text{ even}} x_i \cdot 0 + \sum_{i \text{ odd}} 0 \cdot y_i = 0.$$

Hence $x \in M^\perp$, and we have shown that $M_e \subset M^\perp$.
Now assume that $x \in M^\perp$. We can decompose x into the sum

$$x = (x_1, 0, x_3, 0, x_5, ...) + (0, x_2, 0, x_4, 0, x_6, ...) := x_o + x_e,$$

where $x_o \in M$ and $x_e \in M_e$. Since we assume that $x \in M^\perp$, we know that

$$0 = \langle x, x_o\rangle = \langle x_o + x_e, x_o\rangle = \langle x_o, x_o\rangle + \langle x_e, x_o\rangle = \|x_o\|^2,$$

where we have used that $\langle x_e, x_o\rangle = 0$, as is clear by inspection. But this means that $x_o = 0$, so $x = x_e \in M_e$. Thus $M^\perp \subset M_e$.

5 Let c_f be the subspace of ℓ^2 that consists of all sequences with finitely many non-zero terms.

a) Show that best approximation fails for c_f.

b) Why does this not contradict the best approximation theorem from class?

Solution. a) Let $x = (1, 1/2, 1/3, ...)$. We start by showing that

$$\inf\{\|x - m\| : m \in M\} = 0.$$

Since the norm is always non-negative, we have that

$$\inf\{\|x - m\| : m \in M\} \geq 0.$$

We will show equality by constructing a sequence $\{m_n\}_{n\in\mathbb{N}}$ in M such that $\|x - m_n\| \to 0$. Let

$$m_n = (1, 1/2, 1/3, ..., 1/n, 0, 0, ...).$$

We have

$$\begin{aligned}
\lim_{n\to\infty} \|x - m_n\| &= \lim_{n\to\infty} \|(0, 0, ..., 1/(n+1), 1/(n+2), 1/(n+3), ...)\| \\
&= \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{1}{k^2} \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} - \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} \\
&= \|x\| - \|x\| \\
&= 0
\end{aligned}$$

which was what we needed to prove. Now, suppose there exist a sequence $m \in M$ such that $\|x - m\| = 0$. Then we have $m = x$, but x is not in M since it has no non-zero terms.

b) This does not contradict the best approximation theorem because the conditions for the theorem are not satisfied. Indeed, the subspace M is not closed. To see this, observe that that the sequence $\{m_n\}_{n\in\mathbb{N}}$ converges to x, but x is not in M.