Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods

Fall 2017

Exercise set 11

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let $L^2[-1,1]$ be the closure of $C[-1,1]$ with respect to the innerproduct

$$\langle f,g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}dt.$$

Apply Gram-Schmidt to the monomial basis $\{1, x, x^2, x^3, ...\}$ up to degree 3.

Solution. We follow the procedure.

$$y_1 = 1 \qquad e_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1^2\,dx}} = \frac{1}{\sqrt{2}}$$

$$y_2 = x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{2}}x\,dx = x \qquad e_2 = \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} = \frac{x}{\sqrt{2/3}}$$

$$y_3 = x - \frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{1}{\sqrt{2}}x^2\,dx - \frac{x}{\sqrt{2/3}}\int_{-1}^{1}\frac{x^3}{\sqrt{2/3}}$$

$$= x^2 - \frac{1}{3} \qquad e_3 = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}(x^2-\frac{1}{3})^2\,dx}} = \frac{x^2-\frac{1}{3}}{\sqrt{\frac{8}{45}}}$$

2 Consider the exponential basis $\{e^{2\pi int} : n \in \mathbb{Z}\}$ in $(L^2[0,1], \langle .,.\rangle)$. Verify Parseval's relation for this particular case directly. Try to explain how Fourier series and some of their properties fit into this problem.

Solution. We will first show that the exponential basis is orthonormal. We have

$$\begin{aligned}
\langle e^{2\pi int}, e^{2\pi int}\rangle &= \int_0^1 e^{2\pi int}\,\overline{e^{2\pi int}}\,dt \\
&= \int_0^1 e^{2\pi int}\,e^{-2\pi int}\,dt \\
&= \int_0^1 e^{2\pi i(n-n)t}\,dt \\
&= \int_0^1 e^0\,dt \\
&= \int_0^1 1\,dt \\
&= 1.
\end{aligned}$$

Suppose $n \neq m$. We have

$$\begin{aligned}
\langle e^{2\pi int}, e^{2\pi imt}\rangle &= \int_0^1 e^{2\pi int}\,\overline{e^{2\pi imt}}\,dt \\
&= \int_0^1 e^{2\pi int}\,e^{-2\pi imt}\,dt \\
&= \int_0^1 e^{2\pi i(n-m)t}\,dt \\
&= \left.\frac{e^{2\pi i(n-m)t}}{2\pi i(n-m)}\right|_0^1 \\
&= \frac{e^{2\pi i(n-m)} - e^0}{2\pi i(n-m)} \\
&= \frac{1-1}{2\pi i(n-m)} && (e^{2\pi i(n-m)} = 1 \text{ because } n-m \in \mathbb{Z}) \\
&= 0,
\end{aligned}$$

hence the basis is orthonormal. Parseval's relation says that if $x = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int}$ and $y = \sum_{m=-\infty}^{\infty} b_m e^{2\pi imt}$, then the equation

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

holds. Indeed, we have

$$\begin{aligned}
\langle x, y\rangle &= \langle \sum_{n=-\infty}^{\infty} a_n e^{2\pi int}, \sum_{m=-\infty}^{\infty} b_m e^{2\pi imt}\rangle \\
&= \sum_{n=-\infty}^{\infty} \langle a_n e^{2\pi int}, \sum_{m=-\infty}^{\infty} b_m e^{2\pi imt}\rangle \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle a_n e^{2\pi int}, b_m e^{2\pi imt}\rangle \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n \overline{b_n} \langle e^{2\pi int}, e^{2\pi imt}\rangle \\
&= \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \langle e^{2\pi int}, e^{2\pi int}\rangle && \begin{pmatrix}\text{The innerproducts are 0 for } m \neq n \\ \text{because the basis is orthonormal.}\end{pmatrix} \\
&= \sum_{n=-\infty}^{\infty} a_n \overline{b_n} && \begin{pmatrix}\text{The innerproducts are all 1} \\ \text{because the basis is orthonormal.}\end{pmatrix}
\end{aligned}$$

How does this relate to Fourier series? As the same suggests, it is just the usual Parseval relation for Fourier series (in a not-very-convincing disguise): If $f, g \in L^2([0,1])$, we saw above that we may write

$$f = \sum_{-\infty}^{\infty} a_n e^{2\pi int}$$
$$g = \sum_{n=-\infty}^{\infty} b_n e^{2\pi int}$$

for some coefficients $\{a_n\}_{n\in\mathbb{Z}}, \{b_n\}_{n\in\mathbb{Z}} \in \ell^2$. This is true since the exponential basis is a basis. We also know from the general theory that

$$a_n = \langle f, e^{2\pi int}\rangle = \int_0^1 f(t) e^{-2\pi int}\, dt$$
$$b_n = \langle g, e^{2\pi int}\rangle = \int_0^1 g(t) e^{-2\pi int}\, dt$$

Above we proved that

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b_n},$$

and writing out the inner product explicitly we see that

$$\int_0^1 f(t)\overline{g(t)}\, dt = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

where $a_n = \int_0^1 f(t) e^{-2\pi int}$ and $b_n = \int_0^1 g(t) e^{-2\pi int}$, which is (hopefully) a version of Parseval's relation that you recognize from the theory of Fourier series.

3 We define the cyclic shift matrix T_1 and the modulation matrix M_1 by

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}.$$

a) Show that

1. $T_1 M_1 = e^{2\pi i/3} M_1 T_1$.
2. $T_1^3 = I_3$ and $M_1^3 = I_3$.
3. M_1 and T_1 are unitary matrices.

b) Show that $\{\frac{1}{\sqrt{3}} M_1^k T_1^l : k, l \in \{1,2,3\}\}$ is an orthonormal basis of the space of complex 3×3 matrices $M_3(\mathbb{C})$ with respect to the innerproduct $\langle A, B \rangle = \operatorname{tr}(AB^*)$.

Solution. a) 1. By multiplying the matrices we find that

$$\begin{aligned} M_1 T_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i/3} \\ e^{4\pi i/3} & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$e^{2\pi i/3} M_1 T_1 = \begin{pmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} e^{2\pi i/3} \\ e^{2\pi i/3} e^{4\pi i/3} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \end{pmatrix}$$

On the other hand

$$\begin{aligned} T_1 M_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} \\ &= \begin{pmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

so $T_1 M_1 = e^{2\pi i/3} M_1 T_1$. 2. To show the other two equalities we calculate

$$T_1^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$T_1^3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly

$$M_1^2 = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{2\pi i/3} & 0\\ 0 & 0 & e^{4\pi i/3}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{2\pi i/3} & 0\\ 0 & 0 & e^{4\pi i/3}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{4\pi i/3} & 0\\ 0 & 0 & e^{8\pi i/3}\end{pmatrix}$$

Hence

$$M_1^3 = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{4\pi i/3} & 0\\ 0 & 0 & e^{8\pi i/3}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{2\pi i/3} & 0\\ 0 & 0 & e^{4\pi i/3}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{6\pi i/3} & 0\\ 0 & 0 & e^{12\pi i/3}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

3. We finally show that T_1 and M_1 are unitary matrices, which means that $T_1^* = T_1^{-1}$ and $M_1^* = M_1^{-1}$. By the definition of conjugate transpose of matrices we find that

$$T_1^* = \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \qquad M_1^* = \begin{pmatrix}1 & 0 & 0\\ 0 & e^{-2\pi i/3} & 0\\ 0 & 0 & e^{-4\pi i/3}\end{pmatrix}, \tag{1}$$

and it is straightforward to use this to show that $T_1^*T_1 = I$ and $M_1^*M_1 = I$. Note that this is also implies that $M_1^* = M_1^2$ and $T_1^* = T_2^2$, since we know by a) that $T_1^3 = M_1^3 = I$.

b) First note that it is enough to show that the set is orthonormal. The dimension of $M_3(\mathbb{C})$ is 9, and we have a total of 9 matrices in our set. If the set is orthonormal it must be linearly independent (show this if you do not remember from earlier courses), and a set of 9 linearly independent vectors in a 9-dimensional space must be a basis. To show that the set is orthonormal, we need to show that

$$\langle \frac{1}{\sqrt{3}}M_1^kT_1^l, \frac{1}{\sqrt{3}}M_1^mT_1^n\rangle = \begin{cases}1 \text{ when k=m and l=n,}\\ 0 \text{ otherwise,}\end{cases}$$

which is a total of $9 \cdot 9 = 81$ inner products – a daunting task! Fortunately it is possible to reduce it to fewer cases; we give one approach here, but many other methods are possible. Since $\langle A, B\rangle = \text{tr}(AB^*)$, we need to find

$$(\frac{1}{\sqrt{3}}M_1^kT_1^l)^*.$$

Using that $(AB)^* = B^*A^*$ we find

$$(\frac{1}{\sqrt{3}}M_1^kT_1^l)^* = \frac{1}{\sqrt{3}}(T_1^*)^l(M_1^*)^k. \tag{2}$$

We have also shown that $\text{tr}(AB) = \text{tr}(BA)$ for matrices A, B, and using this along with the fact that T_1 and M_1 are unitary we get

$$\begin{aligned}
\langle \frac{1}{\sqrt{3}} M_1^k T_1^l, \frac{1}{\sqrt{3}} M_1^m T_1^n \rangle &= \frac{1}{3}\mathrm{tr}((M_1^m T_1^n)^* M_1^k T_1^l) \\
&= \frac{1}{3}\mathrm{tr}((T_1^*)^n (M_1^*)^m M_1^k T_1^l) && \text{by equation (2)} \\
&= \frac{1}{3}\mathrm{tr}(T_1^{-n} M_1^{-m} M_1^k T_1^l) && T_1^* = T_1^{-1} \text{ and } M_1^* = M_1^{-1} \\
&= \frac{1}{3}\mathrm{tr}(M_1^{-m} M_1^k T_1^l T_1^{-n}) && \mathrm{tr}(AB) = \mathrm{tr}(BA) \\
&= \frac{1}{3}\mathrm{tr}(M_1^{k-m} T_1^{l-n}).
\end{aligned}$$

If $m = k$ and $l = n$, then the last expression is $\frac{1}{3}\mathrm{tr}(I) = \frac{1}{3}3 = 1$. Hence

$$\langle \frac{1}{\sqrt{3}} M_1^k T_1^l, \frac{1}{\sqrt{3}} M_1^k T_1^l \rangle = 1,$$

as we hoped to show.
To consider the other cases we need to figure out the structure of $\mathrm{tr}(M_1^{k-m} T_1^{l-n})$ more closely, and a natural first step is to get rid of possible negative exponents in $M_1^{k-m} T_1^{l-n}$.
Claim: $M_1^{k-m} T_1^{l-n} = M_1^p T_1^q$ for some $p, q = 0, 1, 2$.
Since $k, m = 1, 2$ or 3, $M_1^{k-m} = M_1^{p'}$ for p' either $-2, -1, 0, 1$ or 2. If $p' = 0, 1, 2$ we have the result we wanted. In the other cases we use that $M_1^* = M_1^2$ to find

$$\begin{aligned}
&M_1^{-1} = M_1^* = M_1^2 \\
&M_1^{-2} = (M_1^{-1})^2 = (M_1^2)^2 = M_1^4 = M_1,
\end{aligned}$$

where we have also used that $M_1^3 = I$. Thus we see that $M_1^{k-m} = M_1^p$ for $p = 0, 1, 2$ in all the cases. The proof that the same is true for T_1^{l-n} is exactly the same.

It remains to show that $\mathrm{tr}(M_1^{k-m} T_1^{l-n}) = 0$ for $k \neq m$ and $l \neq n$, which by the claim above is equivalent to showing that $\mathrm{tr}(M_1^p T_1^q) = 0$ when either $p \neq 0$ or $q \neq 0$. We divide the proof into three cases:

1. $\mathbf{p = 0, q \neq 0}$: In this case we consider $\mathrm{tr}(T_1^q)$ for $q = 1, 2$. Since

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = T_1^2 \qquad T_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3}$$

   we clearly have $\mathrm{tr}(T_1) = \mathrm{tr}(T_1^2) = 0$.

2. $\mathbf{q = 0, p \neq 0}$: In this case we consider $\mathrm{tr}(M_1^p)$ for $p = 1, 2$. Since

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} = T_1^2 \qquad M_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{8\pi i/3} \end{pmatrix}, \tag{4}$$

   we have $\mathrm{tr}(M_1) = 1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$ and $\mathrm{tr}(M_1^2) = 1 + e^{4\pi i/3} + e^{8\pi i/3} = 0$, which may be shown using a geometric sum.

3. $\mathbf{p}, \mathbf{q} \neq \mathbf{0}$: By using the relation $T_1 M_1 = e^{2\pi i/3} M_1 T_1$ several times, we get that

$$T_1^q M_1^p = e^{pq2\pi i/3} M_1^p T_1^q.$$

Since $p, q = 1, 2$, we know that $e^{pq2\pi i/3} \neq 1$ (Clearly $e^{pq2\pi i/3} = 1$ if and only if pq is a multiple of 3). But using the relation $\text{tr}(AB) = \text{tr}(BA)$ we get

$$\text{tr}(M_1^p T_1^q) = \text{tr}(T_1^q M_1^p) = \text{tr}(e^{pq2\pi i/3} M_1^p T_1^q) = e^{pq2\pi i/3}\text{tr}(M_1^p T_1^q),$$

which implies that $\text{tr}(M_1^p T_1^q) = 0$.

4 Let $\{e_n : n \in \mathbb{N}\}$ be the standard basis in the ℓ^p-spaces.

a) Show that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^p for $1 \leq p < \infty$ if and only if $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$.

b) Show that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞ if and only if $(\alpha_n)_{n\in\mathbb{N}}$ converges to zero.

Solution. a) Recall what it means for a series to converge in a normed space such as ℓ_p: we need the partial sums

$$s_N = \sum_{n=0}^{N} \alpha_n e_n$$

to form a convergent sequence in ℓ^p. Assume first that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^p to some sequence $x = (x_1, x_2, ...) \in \ell^p$. As we have seen before, if a sequence y_n converges to some y in a normed space, then $\|y_n\|$ converges to $\|y\|$. In this case we know that $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converges to $x \in \ell^p$, so

$$\|s_N\|_{\ell^p}^p = \|\sum_{n=0}^{N} \alpha_n e_n\|_{\ell^p}^p = \sum_{n=0}^{N} |\alpha_n|^p \to \|x\|_{\ell^p}^p.$$

But this means that the partial sums $\sum_{n=0}^{N} |\alpha_n|^p$ converge to the number $\|x\|_{\ell^p}^p$, which shows that the sum $\sum_{n=0}^{\infty} |\alpha_n|^p$ converges and hence $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$.

Conversely, assume that $(\alpha_n)_{n\in\mathbb{N}} \in \ell^p$. We want to show that the sum $\sum_{n=0}^{\infty} \alpha_n e_n$ converges to the sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ in ℓ^p. As discussed, this means that we need to show that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to α in ℓ^p. Note that

$$\alpha - s_N = (0, 0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...),$$

and therefore

$$\|\alpha - s_N\|_{\ell^p}^p = \sum_{n=N+1}^{\infty} |\alpha_n|^p.$$

Since $\alpha \in \ell^p$ we know that $\sum_{n=0}^{\infty} |\alpha_n|^p$ converges, and as we have used in many previous problems sets this implies that $\sum_{n=N+1}^{\infty} |\alpha_n|^p \to 0$ as $N \to \infty$. Hence

$\|\alpha - s_N\|_{\ell^p}^p \to 0$, so the partial sums s_N converge to α in ℓ^p and this means by definition that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges to α in ℓ^p.

b) Start by assuming that $\alpha_n \to 0$. To show that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞, we will show that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to α. Clearly

$$\alpha - s_N = (\alpha_1, \alpha_2, ...) - (\alpha_1, \alpha_2, ...\alpha_N, 0, 0, ...) = (0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...)$$

so

$$\|\alpha - S_N\|_\infty = \|(0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2}, ...)\|_\infty = \sup_{n \geq N+1} |\alpha_n|.$$

Since $\alpha_n \to 0$, we know that $\sup_{n \geq N+1} |\alpha_n| \to 0$ as $N \to \infty$ (Do make sure that you understand why this is true!).

Conversely, assume that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges in ℓ^∞. This means that the partial sums $s_N = \sum_{n=0}^{N} \alpha_n e_n$ converge to some element $x = (x_1, x_2, ...)$ in ℓ^∞. It is simple to see that $\alpha_i = x_i$ for each $i \in \mathbb{N}$: If $N > i$, then

$$x - s_N = (x_1 - \alpha_1, x_2 - \alpha_2, ..., x_i - \alpha_i, ..., x_N - \alpha_N, x_{N+1}, x_{N+2}, ...),$$

and so

$$|x_i - \alpha_i| \leq \sup_{j \in N} |x_j - s_N(j)| = \|x - s_N\|_\infty \to 0 \text{ as } N \to \infty$$

which implies that $x_i = \alpha_i$. Since we assumed that $s_N \to x$, we now know that $s_N \to \alpha$ in ℓ^∞. Note that

$$\alpha - s_N = (0, 0, ..., 0, \alpha_{N+1}, \alpha_{N+2} ...)$$

so

$$\|\alpha - s_N\|_\infty = \sup_{j > N} |\alpha_j|,$$

and this last expression converges to 0 since $s_N \to \alpha$. This implies that

$$|\alpha_{N+1}| \leq \sup_{j > N} |\alpha_j| = \|\alpha - s_N\|_\infty \to 0 \text{ as } N \to \infty,$$

hence $\alpha_i \to 0$.