Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

$\boxed{1}$ Let

$$z_1 = \sqrt{\frac{2}{3}}\begin{bmatrix}1\\0\end{bmatrix}, \quad z_2 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\\sqrt{3}/2\end{bmatrix}, \quad z_3 = \sqrt{\frac{2}{3}}\begin{bmatrix}-1/2\\-\sqrt{3}/2\end{bmatrix}.$$

Show that for every $x \in \mathbb{R}^2$ we have

a)

$$\|x\|^2 = \sum_{i=1}^{3} |\langle x, z_i\rangle|^2$$

b)

$$x = \sum_{i=1}^{3} \langle x, z_i\rangle z_i$$

Remark. The vectors z_1, z_2, z_3 span $\mathbb{R}^2$, but they are obviously not an orthonormal basis (they are not even linearly independent). Still, they satisfy a generalization of Parseval's identity and "act like" an orthonormal basis. Such systems appear very naturally in applications (e.g. in signal analysis), and are often called Parseval frames. This concrete system is known as the Mercedes Benz frame (can you think of why?).

Solution. **a)** Let $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ be an arbitrary element of $\mathbb{R}^2$.

$$\begin{aligned}
\sum_{i=1}^{3} |\langle x, z_i\rangle|^2 &= \frac{2}{3}\left(x_1^2 + \left(-\frac{x_1}{2} + \frac{\sqrt{3}x_2}{2}\right)^2 + \left(-\frac{x_1}{2} - \frac{\sqrt{3}x_2}{2}\right)^2\right) \\
&= \frac{2}{3}\left(x_1^2 + \frac{x_1^2}{4} + \frac{3x_2^2}{4} - \frac{\sqrt{3}x_1x_2}{2} + \frac{x_1^2}{4} + \frac{3x_2^2}{4} + \frac{\sqrt{3}x_1x_2}{2}\right) \\
&= \frac{2}{3}\left(\frac{3}{2}x_1^2 + \frac{3}{2}x_2^2\right) \\
&= x_1^2 + x_2^2 = \|x\|^2.
\end{aligned}$$

b) We could show this by direct computation, but here we will use a different approach that is more "informative". It turns out that the Parseval identity in a) implies the representation in b). Indeed, for all $x, y \in \mathbb{R}^2$ we have, by a) and the polarization identity

$$\begin{aligned}\langle x, y\rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)\\ &= \frac{1}{4}\sum_{i=1}^{3}\left(\langle x+y, z_i\rangle^2 - \langle x-y, z_i\rangle^2\right)\\ &= \frac{1}{4}\sum_{i=1}^{3}\left((\langle x, z_i\rangle + \langle y, z_i\rangle)^2 - (\langle x, z_i\rangle - \langle y, z_i\rangle)^2\right).\end{aligned}$$

After multiplying out and cancelling terms, we are left with

$$\langle x, y\rangle = \sum_{i=1}^{3}\langle x, z_i\rangle\langle y, z_i\rangle = \langle\sum_{i=1}^{3}\langle x, z_i\rangle z_i, y\rangle$$

for all $x, y \in \mathbb{R}^2$. In particular, this implies that

$$x = \sum_{i=1}^{3}\langle x, z_i\rangle z_i,$$

since the previous calculation shows that $\langle x - \sum_{i=1}^{3}\langle x, z_i\rangle z_i, y\rangle = 0$ for all y, meaning that $x - \sum_{i=1}^{3}\langle x, z_i\rangle z_i \in (\mathbb{R}^2)^{\perp} = \{0\}$ – hence $x - \sum_{i=1}^{3}\langle x, z_i\rangle z_i = 0$ and $x = \sum_{i=1}^{3}\langle x, z_i\rangle z_i$. If you draw these three vectors, they will look kind of like the logo of Mercedes Benz.

$\boxed{2}$ Let $\mathcal{P}_3$ be the space of polynomials of degree at most 3.

a) Show that $\{B_0^3(x) = (1-x)^3, B_1^3(x) = 3x(1-x)^2, B_2^3(x) = 3x^2(1-x), B_3^3(x) = x^3\}$ is a basis for $\mathcal{P}_3$, known as the Bernstein basis. Try to define a Bernstein basis for the space of polynomials of degree at most n.

b) Since $\{B_i^3(x) : i = 0, ..., 3\}$ is a basis of $\mathcal{P}_3$ there exist unique coefficients $\alpha_0, ..., \alpha_3$ for any $f \in \mathcal{P}_3$ such that

$$f(x) = \alpha_0 B_0^3(x) + \alpha_1 B_1^3(x) + \alpha_2 B_2^3(x) + \alpha_3 B_3^3(x).$$

On the other hand we have

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Express α_i in terms of a_i for $i = 0, ..., 3$. In other words, how does one convert a polynomial in monomial form to one in the Bernstein basis?

Solution. a) We start by showing that the set is linearly independent, in other words that

$$\alpha_0 B_0^3 + \alpha_1 B_1^3 + \alpha_2 B_2^3 + \alpha_3 B_3^3 = 0 \quad \Rightarrow \quad \alpha_i = 0 \quad \forall i = 0, \ldots, 3.$$

Assume that

$$\alpha_0 B_0^3 + \alpha_1 B_1^3 + \alpha_2 B_2^3 + \alpha_3 B_3^3 = 0.$$

Replacing the B_i^n's with their definition, we get

$$\alpha_0(1-x)^3 + 3\alpha_1 x(1-x)^2 + 3\alpha_2 x^2(1-x) + \alpha_3 x^3 = 0.$$

By multiplying out the brackets and rearranging we get

$$(\alpha_3 - 3\alpha_2 + 3\alpha_1 - \alpha_0)x^3 + (3\alpha_2 - 6\alpha_1 + 3\alpha_0)x^2 + (3\alpha_1 - 3\alpha_0)x + \alpha_0 = 0,$$

which means that the coefficients are all 0:

$$\begin{aligned} \alpha_3 - 3\alpha_2 + 3\alpha_1 - \alpha_0 &= 0 \\ 3\alpha_2 - 6\alpha_1 + 3\alpha_0 &= 0 \\ 3\alpha_1 - 3\alpha_0 &= 0 \\ \alpha_0 &= 0 \end{aligned}$$

By repeatedly substituting from bottom and up, we get that $a_0 = a_1 = a_2 = a_3 = 0$. Hence the set is linearly independent. To show that it is a basis, it is enough to observe that the set has 4 elements, which is the same as the dimension of the vector space $\mathcal{P}_3$.

The Bernstein basis for $\mathcal{P}_n$ is given by $\{B_i^n\}_{i=1}^n$, where

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

b) We have

$$\alpha_0 B_0^3 + \alpha_1 B_1^3 + \alpha_2 B_2^3 + \alpha_3 B_3^3 = a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

By rearranging the left-hand side as we did in a) we get

$$(\alpha_3 - 3\alpha_2 + 3\alpha_1 - \alpha_0)x^3 + (3\alpha_2 - 6\alpha_1 + 3\alpha_0)x^2 + (3\alpha_1 - 3\alpha_0)x + \alpha_0 = a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

Since the coefficients have to match up, we get

$$\begin{aligned} a_0 &= \alpha_0 \\ a_1 &= 3\alpha_1 - 3\alpha_0 \\ a_2 &= 3\alpha_2 - 6\alpha_1 + 3\alpha_0 \\ a_3 &= \alpha_3 - 3\alpha_2 + 3\alpha_1 - \alpha_0, \end{aligned}$$

and after solving for the α_i's we get

$$\begin{aligned} \alpha_0 &= a_0 \\ \alpha_1 &= \frac{a_1}{3} + a_0 \\ \alpha_2 &= \frac{a_2}{3} + \frac{2a_1}{3} + a_0 \\ \alpha_3 &= a_3 + a_2 + a_1 + a_0 \end{aligned}$$

$\boxed{3}$ Let X be a vector space and $\mathcal{B} = \{b_1, b_2, ..., b_n\}$ be a basis for X. Suppose $T : X \to X$ is a linear mapping. Prove that the following statements are equivalent:

1. The matrix representing T has upper-triangular form.
2. $Tb_j \in \text{span}(b_1, ..., b_j)$ for $j = 1, ..., n$.
3. $\text{span}(b_1, ..., b_j)$ is invariant under T for $j = 1, ..., n$.

Solution. **2**. $\implies$ **3**. : We assume that $Tb_j \in \text{span}(b_1, ..., b_j)$ for $j = 1, ..., n$, and need to show that if $x \in \text{span}(b_1, ..., b_k)$ for some k, then $Tx \in \text{span}(b_1, ..., b_k)$. Since $x \in \text{span}(b_1, ..., b_k)$ we may find scalars $\lambda_1, ..., \lambda_k$ such that

$$x = \sum_{i=1}^{k} \lambda_i b_i.$$

Since T is linear we get that

$$Tx = \sum_{i=1}^{k} \lambda_i T b_i. \tag{1}$$

By assumption, $Tb_i \in \text{span}(b_1, ..., b_k)$ for $i = 1, 2, ..., k$[1]. Hence equation (1) expresses Tx as a linear combination of elements in the subspace $\text{span}(b_1, ..., b_k)$, and since a subspace is closed under linear combinations we may conclude that $Tx \in \text{span}(b_1, ..., b_k)$.
3. $\implies$ **2**. : Now we assume that $Tx \in \text{span}(b_1, ..., b_j)$ whenever $x \in \text{span}(b_1, ..., b_j)$ for $j = 1, ..., n$. We need to show that $Tb_k \in \text{span}(b_1, ..., b_k)$ for $k = 1, ..., n$. Clearly $b_k \in \text{span}(b_1, ..., b_k)$, so it follows straight from our assumption that $Tb_k \in \text{span}(b_1, ..., b_k)$.
1. $\implies$ **2**. : Recall that the matrix $A = (a_{ij})$ of T with respect to $\mathcal{B}$ is defined by

$$Tb_j = \sum_{i=1}^{n} a_{ij} b_i. \tag{2}$$

If A is upper triangular, we know that $a_{ij} = 0$ for $i > j$. Equation (2) then says that

$$Tb_j = \sum_{i=1}^{j} a_{ij} b_i,$$

hence $Tb_j \in \text{span}(b_1, ..., b_j)$.
2. $\implies$ **1**. : If we assume that $Tb_j \in \text{span}(b_1, ..., b_j)$ for $j = 1, ..., n$, we may for each $j = 1, ..., n$ find scalars a_{ij} such that

$$Tb_j = \sum_{i=1}^{j} a_{ij} b_i.$$

However, by comparing this to the definition of the matrix representation $A = (a_{ij})$ of T in equation (2) we see that this implies $a_{ij} = 0$ for $i > j$ – so the matrix representation A is upper triangular.

[1] We even have the stronger condition that $Tb_i \in \text{span}(b_1, ..., b_i)$, but clearly $\text{span}(b_1, ..., b_i) \subset \text{span}(b_1, ..., b_k)$ when $i \leq k$.

4 Let X be a vector space of dimension n. Suppose $\{b_1, b_2, ..., b_k\}$ is a linearly independent subset of X for some $k < n$. Show that there exist vectors $b_{k+1}, ..., b_n$ such that $\{b_1, b_2, ..., b_n\}$ is a basis for X.

Solution. We assume that $\{b_1, b_2, ..., b_k\}$ is a linearly independent subset of X. If $\text{span}(b_1, b_2, ..., b_k) = X$, then $\{b_1, b_2, ..., b_k\}$ would be a basis for X with $k < n$ elements. This is impossible, since the dimension of X is n. Hence $\text{span}(b_1, b_2, ..., b_k) \neq X$, so let us pick some $b_{k+1} \notin \text{span}(b_1, b_2, ..., b_k)$. We claim that the set $\{b_1, b_2, ..., b_k, b_{k+1}\}$ is still linearly independent: assume

$$\sum_{i=1}^{k+1} \lambda_i b_i = 0 \tag{3}$$

for scalars λ_i. If $\lambda_{k+1} \neq 0$, we may rearrange equation (3) to get

$$b_{k+1} = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1}} b_i$$

which implies $b_{k+1} \in \text{span}(b_1, b_2, ..., b_k)$ – a contradiction. Thus $\lambda_{k+1} = 0$, and then

$$\sum_{i=1}^{k} \lambda_i b_i = 0,$$

hence $\lambda_i = 0$ for $i = 1, 2, ..., k+1$ since $\{b_1, b_2, ..., b_k\}$ is linearly dependent. To summarize this, we have seen that a set of k linearly independent vectors may be extended to a set of $k+1$ linearly independent vectors, as long as $k < n$. We may therefore repeat this procedure: starting from the linearly independent set $\{b_1, b_2, ..., b_k\}$, we obtain first a linearly independent set $\{b_1, b_2, ..., b_k, b_{k+1}\}$, then a linearly independent set $\{b_1, b_2, ..., b_k, b_{k+1}, b_{k+2}\}$, etc., until we finally obtain a linearly independent set $\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n\}$ [2]. The proof is complete if we can show that $\text{span}\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n\} = X$.
Let x be any vector in X. The set $\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n, x\}$ has $n+1$ elements, and we know from linear algebra (or lemma 7.3 in the notes) that a set of vectors with more elements than the dimension of the space is linearly dependent. Hence $\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n, x\}$ is linearly dependent. By definition, this means that there exist scalars λ_i for $i = 1, 2, ..., n+1$ such that

$$\lambda_1 b_1 + \lambda_2 b_2 + ... + \lambda_n b_n + \lambda_{n+1} x = 0 \tag{4}$$

and not all λ_i are zero. If we had $\lambda_{n+1} = 0$ we get

$$\lambda_1 b_1 + \lambda_2 b_2 + ... + \lambda_n b_n = 0,$$

which by the linear independence of $\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n\}$ means that $\lambda_i = 0$ for $i = 1, 2, ..., n$. This contradicts the fact that some $\lambda_i \neq 0$. Hence we must have that $\lambda_{n+1} \neq 0$. In this case we may rearrange equation (4) to get

$$x = \frac{\lambda_1}{\lambda_{n+1}} b_1 + \frac{\lambda_2}{\lambda_{n+1}} b_2 + ... + \frac{\lambda_n}{\lambda_{n+1}} b_n,$$

[2] At this point we may not proceed further, our method of extension depended on $k < n$.

which shows that $x \in \text{span}\{b_1, b_2, ..., b_n\}$. Since x was an arbitrary element in X, this means that $\text{span}\{b_1, b_2, ..., b_k, b_{k+1}, ..., b_n\} = X$, which is what we wanted to show.