Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4145 Linear Methods
Fall 2017

Exercise set 13

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Given the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

a) Compute the singular value decomposition of A.

b) Use the result of a) to find:

1. Bases for the following vector spaces: $\ker(A), \ker(A^*), \operatorname{ran}(A), \operatorname{ran}(A^*)$.
2. The pseudo-inverse of A.
3. Find the minimal norm solution of $Ax = b$ for $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution. a) We need to find matrices U, V, Σ with certain properties such that $A = U\Sigma V^*$. If we look back to the proof of the SVD in the lecture notes (theorem 7.12), we can deduce how we construct U, V and Σ. Note that since A is a 3×2-matrix with rank 2, we will find that V is a 2×2 matrix, Σ a 3×2-matrix and U a 3×3-matrix.

1. V is picked as the matrix that diagonalizes A^*A, and therefore the columns of U are the normalized eigenvectors of A^*A.
2. Σ is the 3×2 matrix with the positive singular values of A (= the square roots of the positive eigenvalues of A^*A) along the diagonal, and zero elsewhere. [1]
3. The first 2 columns of the 3×3 U will be the vectors Av_1 and Av_2, where v_1, v_2 are the columns of V. Since we need U to be a 3×3 matrix, we need one more column v_3, which we find by picking a normalized vector v_3 that is orthogonal to v_1 and v_2.

[1] If we did not have enough singular values to fill the diagonal of Σ, we would just put zeros in the rest of the diagonal of Σ. This will not be an issue in this example.

Let us now find these matrices.

1. $A^*A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}$ has as characteristic polynomial $x^2 - 18x + 17 = (x-17)(x-1)$. Hence the eigenvalues of A^*A are $\lambda_1 = 17$ and $\lambda_2 = 1$. The corresponding normalized eigenvectors are $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Consequently, we have
$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

2. The singular values of A are $\sigma_1 = \sqrt{17}$ and $\sigma_2 = 1$. Thus
$$\Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3. The first two columns of U are given by
$$u_1 = \frac{1}{\sqrt{17}}\frac{1}{\sqrt{2}}Av_1 = \frac{1}{\sqrt{34}}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}}\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$$
and by
$$u_2 = \frac{1}{1}Av_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$
Consequently, U has the form
$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & x_1 \\ \frac{4}{\sqrt{34}} & 0 & x_2 \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & x_3 \end{pmatrix}$$
The last column is determined by the assumption that it has to be orthogonal to the first two columns. The choice
$$u_3 = \frac{1}{\sqrt{17}}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$
satisfies these conditions, but there are many other ways to complete the first two columns to become an orthonormal basis for $\mathbb{C}^3$,
$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix}.$$

4. The SVD of A is

$$\begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{-1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix} \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

b)

1. This follows from proposition 7.4.3 in the notes (recall that $r = 2$ in our case, by inspection):

   $$\ker(A) = \{0\}, \ker(A^*) = \operatorname{span}(u_3), \operatorname{ran}(A) = \operatorname{span}(u_1, u_2), \operatorname{ran}(A^*) = \operatorname{span}(v_1, v_2).$$

2. By the discussion on page 104 in the notes, $A^\dagger = V\Sigma^+U^*$ where Σ^+ is the matrix obtained from Σ by replacing the singular values σ_i with σ_i^{-1} and taking the transpose. Hence

   $$A^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{17}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{34}} & \frac{4}{\sqrt{34}} & \frac{3}{\sqrt{34}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{17}} & \frac{-3}{\sqrt{17}} & \frac{2}{\sqrt{17}} \end{pmatrix} = \begin{pmatrix} \frac{-7}{17} & \frac{2}{17} & \frac{10}{17} \\ \frac{10}{17} & \frac{2}{17} & \frac{-7}{17} \end{pmatrix}.$$

3. By the discussion on page 104 of the notes, the least squares solution is given by

   $$A^\dagger b = \begin{pmatrix} \frac{-7}{17} & \frac{2}{17} & \frac{10}{17} \\ \frac{10}{17} & \frac{2}{17} & \frac{-7}{17} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{27}{17} \\ \frac{-7}{17} \end{pmatrix}.$$

2 Let U be a $n \times n$ matrix with columns $u_1, ..., u_n$. Show that the following statements are equivalent:

1. U is unitary.
2. $\{u_1, ..., u_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

Solution. Note that the column u_i is of the form $u_i = (u_{1,i}, u_{2,i}, \dots, u_{n,i})^T$, and the inner product between two columns is given by

$$\begin{aligned} \langle u_i, u_j \rangle &= \sum_{k=1}^{n} u_{k,i}\overline{u_{k,j}} \\ &= u_i \cdot u_j^*, \end{aligned}$$

where $u_i \cdot u_j^*$ is the usual dot product for vectors in $\mathbb{C}^n$ and

$$u_j^* = \begin{pmatrix} \overline{u_{1,j}} \\ \overline{u_{2,j}} \\ \dots \\ \overline{u_{n,j}} \end{pmatrix}.$$

1 ⇒ 2

Assume that $U^*U = I$. Recall that element (i, j) of the matrix product U^*U is the dot product of row i of U^* with column j of U. Since row i of U^* is u_i^*, this means that element (i, j) of U^*U is $u_i^* \cdot u_j$. Furthermore $U^*U = I$, which implies that $u_i^* u_j = \delta_{i,j}$ for $i, j = 1, \dots, n$. Then we have

$$\langle u_j, u_i \rangle = u_i^* \cdot u_j = \delta_{i,j},$$

hence $(u_1, u_2, \dots, u_n)$ is an orthonormal system of vectors in $\mathbb{C}^n$. To show that it is a basis for $\mathbb{C}^n$ it is enough to note that $\mathbb{C}^n$ has dimension n, and the system consists of n vectors. Hence the columns form a linearly independent subset of n vectors in an n-dimensional space, and it follows that the columns form a basis.

2 ⇒ 1

Assume that the columns $u_1, u_2, \dots, u_n$ of U are an orthonormal basis of $\mathbb{C}^n$, i.e.

$$\langle u_i, u_j \rangle = \delta_{i,j},$$

for $i, j = 1, \dots, n$. Then we have

$$u_i \cdot u_j^* = \langle u_i, u_j \rangle = \delta_{i,j},$$

hence we have $U^*U = I$. One gets that $U^*U = I$ from exactly the same argument, or by knowing that a left inverse of matrix is also a two-sided inverse.

3 Let T be the shift operator on ℓ^2 defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$.

1. Show that T has no eigenvalues.
2. Does T^* have any eigenvalues?

Solution.

1. Assume that λ is an eigenvalue of T with eigenvector $y = (y_1, y_2, \dots)$. Then $Ty = \lambda y$, and writing out both sides we find

$$(0, y_1, y_2, \dots) = (\lambda y_1, \lambda y_2, \lambda y_3 \dots). \tag{1}$$

In particular $\lambda y_1 = 0$, which implies that either $y_1 = 0$ or $\lambda = 0$. If $\lambda = 0$, then equality (1) becomes

$$(0, y_1, y_2, \dots) = (0, 0, \dots)$$

which shows that $y = 0$, hence not an eigenvector. We may therefore assume that $y_1 = 0$ and $\lambda \neq 0$. In this case the equality (1) becomes

$$(0, 0, y_2, \dots) = (0, \lambda y_2, \lambda y_3 \dots), \tag{2}$$

which implies that $y_2 = 0$. Inserting this back into the equation, we find

$$(0, 0, 0, \dots) = (0, 0, \lambda y_3 \dots), \tag{3}$$

hence $y_3 = 0$. We may clearly continue like this to show that y is 0 in all coordinates, hence $y = 0$ and y is not an eigenvector.

2. We know from the lectures (and it is not difficult to show) that $T^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$. This operator has eigenvalues. For instance, let $y = (1, \frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^{p-1}}, \dots)$. Then
$$T^* y = (\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^p}, \dots) = \frac{1}{2} y,$$
hence y is an eigenvector with eigenvalue $\frac{1}{2}$. Note that $y \in \ell^2$, which we needed since we defined T on ℓ^2.

4 Let X be a finite dimensional vector space and $T : X \to X$ a linear transformation on X. Show that $X = \ker(T) \oplus \operatorname{ran}(T^*)$, where $\oplus$ denotes the direct sum of the vector spaces.

Hint: Use that we know that $\ker(T)^\perp = \operatorname{ran}(T^*)$.

Solution. First note that any finite-dimensional vector space can be made into a Hilbert space: if $\{e_i\}_{i=1}^n$ is a basis for X and $x, y \in X$, then we may write

$$x = \sum_{i=1}^n x_i e_i \qquad\qquad y = \sum_{i=1}^n y_i e_i$$

for coefficients $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$. If we define the inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y_i}$, then X becomes a Hilbert space. Any linear transformation on a finite-dimensional normed space is bounded, and by a previous exercise we then know that $\ker(T)$ is a closed subspace of X. By the projection theorem we get that $X = \ker(T) \oplus \ker(T)^\perp$, and since $\ker(T)^\perp = \operatorname{ran}(T^*)$ we have proved the result.
Note: If we look at the dimensions of $X = \ker(T) \oplus \operatorname{ran}(T^*)$, we see that the dimension of X must be the sum of the dimensions of $\ker(T)$ and $\operatorname{ran}(T^*)$. If we pick a basis to write $T = (T_{ij})$ as an $n \times n$-matrix, this says that

$$n = \dim(\ker(T)) + \dim(\operatorname{ran}(T^*)) = \text{nullity(T)} + \operatorname{rank}(T),$$

the familiar rank-nullity theorem (assuming that we know that $\dim(\operatorname{ran}(T^*)) = \dim(\operatorname{ran}(T))$, which is the statement that the row space and column space of a matrix have the same dimension).