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TMA4145 Linear Methods
Fall 2017

Exercise set 5: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a normed space $(X, \|.\|)$.

a) Show that $(x_n)_{n\in\mathbb{N}}$ is a bounded subset of X.

b) Show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution. a) Denote the limit of $(x_n)_{n\in\mathbb{N}}$ by x. Since the sequence converges, we can find an $N \in \mathbb{N}$ such that $\|x - x_n\| < 1$ for any $n \geq N$. This is simply the definition of convergence in a normed space, with $\epsilon = 1$. For $n \geq N$, we can bound the norm of $\|x_n\|$, since

$$\begin{aligned}\|x_n\| &= \|x_n - x + x\| \\ &\leq \|x_n - x\| + \|x\| \\ &\leq 1 + \|x\|\end{aligned}$$

by the triangle inequality. Hence $1 + \|x\|$ is an upper bound for $\|x_n\|$ for $n \geq N$. Since N is a finite number, we then find an upper bound B for $\|x_n\|$ for *every* $n \in \mathbb{N}$, by defining

$$B = \max(\|x_1\|, \|x_2\|, ..., \|x_{N-1}\|, 1 + \|x\|).$$

b) To show that the sequence is Cauchy we need, for every $\epsilon > 0$, to find $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ whenever $m, n \geq N$. Let us therefore fix some arbitrary $\epsilon > 0$. Since $x_n \to x$, we can find $N \in \mathbb{N}$ such that $\|x - x_n\| < \frac{\epsilon}{2}$ for every $n \geq N$. Then, if $m, n \geq N$, we have

$$\begin{aligned}\|x_m - x_n\| &= \|x_m - x + x - x_n\| \\ &\leq \|x_m - x\| + \|x - x_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

by the triangle inequality. Hence this N works.

2 We denote by c_f the vector space of all sequences with only finitely many non-zero terms. Show that c_f is not a Banach space with the norm $\|\cdot\|_\infty$. As usual, $\|\cdot\|_\infty$ is defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$$

for a sequence $x = (x_n)_{n\in\mathbb{N}} \in c_f$.

Solution. We will find a sequence $y_n \in c_f$ that is Cauchy yet not convergent (note that y_n is a sequence for each value of n – we have a sequence of sequences). Define

$$y_n = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, ...).$$

If we now consider y_n to be elements of the larger space ℓ^∞, we have that $y_n \to y$ where

$$y = (1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{1000}, \frac{1}{1001}, ...).$$

To prove that $y_n \to y$, simply note that $\|y - y_n\|_\infty = \sup_{k\geq n} \frac{1}{k} = \frac{1}{n}$, and clearly $\frac{1}{n} \to 0$. The "problem" in this case is that $y_n \in c_f$, yet clearly $y \notin c_f$. Since the sequence y_n converges, it is Cauchy by problem 1b. However, since the limit is unique and $y_n \to y$ in ℓ^∞, the sequence y_n cannot converge in c_f (if it did converge to some element y' in c_f, the sequence would have two different limits $y \neq y'$ in ℓ^∞).

3 For each $n \in \mathbb{N}$, let

$$x^{(n)} := (1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots),$$

which we regard as an element of the space $\ell^p(\mathbb{R})$ (for any given $p \in [1, \infty]$).

a) Find the limit of the sequence $\left(x^{(n)}\right)_{n\geq 1}$ in $(\ell^\infty(\mathbb{R}), \|\cdot\|_\infty)$. Prove your claim.

b) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$? If the limit exists, find it and prove that it is the limit.

c) Does $\left(x^{(n)}\right)_{n\geq 1}$ have a limit in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$? If the limit exists, find it and prove that it is the limit.

Solution. a) Let $x := \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right)$. Then clearly $x \in \ell^\infty(\mathbb{R})$.

We show that $x^{(n)} \to x$ with respect to the $\|\cdot\|_\infty$ norm. It is enough to show that

$$\|x^{(n)} - x\|_\infty \to 0.$$

But

$$x^{(n)} - x = \left(0, \ldots, 0, -\frac{1}{n+1}, -\frac{1}{n+2}, \ldots\right),$$

so

$$\|x^{(n)} - x\|_\infty = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

b) Let us assume that $\left(x^{(n)}\right)$ has a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$, and let us denote that limit by y.

It follows from lemma 4.1 in the notes that

$$\|x^{(n)}\|_1 \to \|y\|_1.$$

But

$$\|x^{(n)}\|_1 = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \to \infty \quad \text{as } n \to \infty\,,$$

so $\|y\|_1 = \infty$. But then $y \notin \ell^1(\mathbb{R})$, so the sequence $\left(x^{(n)}\right)$ cannot have a limit in $(\ell^1(\mathbb{R}), \|\cdot\|_1)$.

c) We show that the sequence $\left(x^{(n)}\right)$ converges to the vector x defined in part a) also in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$. First note that $x \in \ell^2(\mathbb{R})$, since

$$\|x\|_2^2 = \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty\,.$$

It is enough to show that $\|x^{(n)} - x\|_2 \to 0$. We have:

$$\|x^{(n)} - x\|_2^2 = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots \to 0 \quad \text{as } n \to \infty\,,$$

because this is the *tail* of the convergent series $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j^2}$.

4 Let $C[a, b]$ be the vector space of all continuous functions $f : [a, b] \to \mathbb{R}$.
We will consider two norms on this space, $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

a) Prove that for all $f \in C[a, b]$ we have

$$\|f\|_1 \le (b - a)\, \|f\|_\infty\,.$$

b) Let (f_n) be a sequence in $C[a, b]$.
Prove that if $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $f_n \to f$ with respect to $\|\cdot\|_1$.

c) Show that the reverse of the statement in b) is not always true.

Solution. a) Since for all $x \in [a, b]$, we have $|f(x)| \le \|f\|_\infty$, we derive the following:

$$\|f\|_1 = \int_a^b |f(x)|\, dx \le \int_a^b \|f\|_\infty\, dx = (b - a)\, \|f\|_\infty\,.$$

b) If $f_n \to f$ with respect to $\|\cdot\|_\infty$ then $\|f_n - f\|_\infty \to 0$ – this is more or less the definition of convergence in a normed space.

But from part a) of this problem, $\|f_n - f\|_1 \leq (b-a)\,\|f_n - f\|_\infty$.

Then by the squeeze test, we must also have that $\|f_n - f\|_1 \to 0$, showing, again by problem 1 part a), that $f_n \to f$ with respect to $\|\cdot\|_1$.

c) For simplicity, let us work with $C[0,1]$.

We define $f\colon [0,1] \to \mathbb{R}$ as $f(x) = 1$ for all x.

Moreover, we define the sequence of functions $(f_n)_{n\geq 1}$ as follows:

$$f_n(x) = \begin{cases} nx, & \text{if } x \in [0, \frac{1}{n}] \\ 1, & \text{if } x \in [\frac{1}{n}, 1]\,. \end{cases}$$

Make sure you draw a picture of these functions, it is more important than their actual formulas. An easy calculation then shows that

$$\|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\,dx = \frac{1}{2}\,(1 \times \frac{1}{n}) = \frac{1}{2n} \to 0\,,$$

so $f_n \to f$ with respect to the $\|\cdot\|_1$ norm.

On the other hand, $f(0) = 1$ and for all $n \geq 1$ we have $f_n(0) = 0$, so $|f_n(0) - f(0)| = 1$. This shows that

$$\|f_n - f\|_\infty \geq 1 \quad \text{for all} \;\; n \geq 1\,,$$

hence f_n cannot converge to f with respect to the $\|\cdot\|_\infty$ norm.