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Exercise set 6:Solutions

First a little note regarding the solutions. The solutions to problem 1 and 2 are very detailed, and this level of detail is not necessarily expected of the students. In problem 1, the main focus is that we reduce the problem to the complex numbers, and then argue similarly to how we proved $\mathbb{R}^n$ is complete in class to get completeness for $\mathbb{C}^n$. The only difference is that we need the last problem on problem set 4, since we now deal with the 2-norm rather than the $\infty-$norm.

$\boxed{1}$ Show that $(\mathbb{C}^n, \|.\|_2))$ is complete.
Hint: Recall from problem (5) of problem set 4 that for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, *we have that*

$$\sum_{i=1}^{n} |x_i| \leq n^{1/2} (\sum_{i=1}^{n} |x_i|^2)^{1/2}.$$

This might be useful at some point in your proof.

Solution. Recall that any completeness proof for normed spaces ($\mathbb{R}^n$, ℓ^1...) has been based on the completeness of the real numbers. It is therefore natural to start by showing that $\mathbb{C}$ is complete, and then use this to get that $\mathbb{C}^n$ is complete.
1: $\mathbb{C}$ is complete.

We consider a Cauchy sequence (c_n) of complex numbers. If we write each c_n using the real and imaginary part, we get that $c_n = a_n + ib_n$ for some $a_n, b_n \in \mathbb{R}$, and then we know that $|c_n| = \sqrt{a_n^2 + b_n^2}$. Let us consider the real part of the sequence: a_n. By the hint with $n = 2$, we get that $|\xi| + |\eta| \leq \sqrt{2}\sqrt{|\xi|^2 + |\eta|^2}$ for $\eta, \xi \in \mathbb{R}$. Using this, we find that

$$\begin{aligned} |a_n - a_m| \leq |a_n - a_m| + |b_n - b_m| &\leq \sqrt{2}\sqrt{|a_n - a_m|^2 + |b_n - b_m|^2} \\ &= \sqrt{2}|c_n - c_m|. \end{aligned}$$

Since (c_n) is Cauchy, it therefore follows that (a_n) is a Cauchy sequence of real numbers (if this is not obvious to you, it is actually made rigorous in problem 3!). Since $\mathbb{R}$ is complete, we get that (a_n) converges to some real number a. Similarly the imaginary parts (b_n) will converge to some real number b. To show that $\mathbb{C}$ is complete, it only remains to show that (c_n) converges to $c := a + bi$ in the norm of

$\mathbb{C}$. Using the triangle inequality we calculate that

$$\begin{aligned} |c - c_n| &= |a + ib - (a_n + ib_n)| \\ &= |a - a_n + i(b - b_n)| \\ &\leq |a - a_n| + |b - b_n| \\ &= |a - a_n| + |b - b_n|. \end{aligned}$$

Since the last expression converges to zero as $n \to \infty$, we have shown that $c_n \to c$. This shows that $\mathbb{C}$ is complete.

2: $\mathbb{C}^n$ is complete

Let $(x^{(k)})$ be a Cauchy sequence in $\mathbb{C}^n$, where we use the notation $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)})$. Let us fix the j'th coordinate for some $1 \leq j \leq n$, and consider the sequence $(x_j^{(k)})_k$ of complex numbers. We find that

$$\begin{aligned} |x_j^{(k)} - x_j^{(l)}| &\leq \sum_{i=1}^{n} |x_i^{(k)} - x_i^{(l)}| \\ &\leq \sqrt{n} \sqrt{\sum_{i=1}^{n} |x_i^{(k)} - x_i^{(l)}|^2} \\ &= \sqrt{n} \|x^{(k)} - x^{(l)}\|_{\mathbb{C}^n}, \end{aligned}$$

where the middle step is the hint. Since $(x^{(k)})$ is Cauchy, we get (from problem 3, if you like) that the sequence $(x_j^{(k)})_k$ is a Cauchy sequence of complex numbers, and hence converges to some complex number x_j. This was true for any $1 \leq j \leq n$, so we can form the element $x = (x_1, x_2, ...x_n) \in \mathbb{C}^n$, which we hope is the limit of our sequence $(x^{(k)})$. To show that x is this limit, we consider

$$\|x - x^{(k)}\|_{\mathbb{C}^n} = \sqrt{\sum_{i=1}^{n} \|x_i - x_i^{(k)}\|_{\mathbb{C}}^2}.$$

Since $x_i^{(k)} \to x_i$ as $k \to \infty$ (this is how we defined x_i!), we get that the right hand side converges to 0 as $k \to \infty$. Hence $\|x - x^{(k)}\|_{\mathbb{C}^n} \to 0$ as $k \to \infty$, and $x^{(k)} \to x$ in $\mathbb{C}^n$. This completes the proof.

$\boxed{2}$ Show that $(\ell^2, \|.\|_2)$ is a Banach space.

Solution. We follow the same approach we used in class for proving that $(\ell^1(\mathbb{R}), \|\cdot\|_1)$ was complete. Recall that

$$\ell^2(\mathbb{R}) = \left\{ x = (x_1, x_2, ..., x_j, ...) : \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$$

and if $x = (x_1, x_2, \dots, x_j, \dots) \in \ell^2(\mathbb{R})$, then its norm is defined as

$$\|x\|_2 := \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{1/2} .$$

Let $\left(x^{(n)}\right)_{n\geq 1}$ be a Cauchy sequence in $(\ell^2(\mathbb{R}), \|\cdot\|_2)$. We prove that it must be convergent.

Step 1: We start off by finding a candidate for the limit of $\left(x^{(n)}\right)_{n\geq 1}$ as $n \to \infty$. Note that we have a sequence whose terms are sequences of real numbers. Let us write them down more explicitly.

$$\begin{aligned} x^{(1)} &= (x_1^{(1)},\, x_2^{(1)},\, \dots,\, x_j^{(1)},\, \dots) \\ \\ x^{(2)} &= (x_1^{(2)},\, x_2^{(2)},\, \dots,\, x_j^{(2)},\, \dots) \\ &\vdots \\ x^{(n)} &= (x_1^{(n)},\, x_2^{(n)},\, \dots,\, x_j^{(n)},\, \dots) \\ &\vdots \end{aligned}$$

Then it is easy to see that since $\left(x^{(n)}\right)_{n\geq 1}$ is Cauchy with respect to the $\|\cdot\|_2$ norm, then the sequence of real numbers formed by picking the first term in each of the vectors $x^{(1)}$, $x^{(2)}$, $\dots x^{(n)}$, $\dots$ we also get a Cauchy sequence; then doing the same but with the second terms, we get a Cauchy sequence in $\mathbb{R}$; $\dots$ doing the same with the j-th terms, we get a Cauchy sequence in $\mathbb{R}$, etc.

Indeed, every vector $x = (x_1, x_2, \dots, x_j, \dots) \in \ell^2(\mathbb{R})$ satisfies the following property:

$$|x_j| \leq \|x\|_2 \quad \text{for all indices } j.$$

That is because

$$|x_j| = \left(|x_j|^2\right)^{1/2} \leq \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} = \|x\|_2.$$

Then for every index j we have

$$|x_j^{(n)} - x_j^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_2 \to 0 \quad \text{as } n, m \to \infty,$$

showing that $\left(x_j^{(n)}\right)_{n\geq 1}$ is a Cauchy of *real numbers*. But $\mathbb{R}$ is complete (which is the crucial fact we use to prove the completeness of most other subspaces). Therefore, this sequence has a limit.

For every index j, let then

$$z_j := \lim_{n\to\infty} x_j^{(n)} ,$$

and define

$$z := (z_1, z_2, \ldots, z_j, \ldots) \,,$$

which is our candidate for the limit of $\left(x^{(n)}\right)_{n\geq 1}$.

Step 2: Before everything else, we need to make sure that the vector z thus constructed is an element of the space $\ell^2(\mathbb{R})$.

Since the sequence $\left(x^{(n)}\right)_{n\geq 1}$ is Cauchy, it is *bounded* with respect to the $\|\cdot\|_2$ norm. That is, there is a constant $C \in \mathbb{R}$ such that

$$\|x^{(n)}\|_2 \leq C \quad \text{for all } \; n \geq 1 \,,$$

hence

$$\|x^{(n)}\|_2^2 \leq C^2 \quad \text{for all } \; n \geq 1 \,.$$

But

$$\|x^{(n)}\|_2^2 = \sum_{j=1}^{\infty} |x_j^{(n)}|^2 \,,$$

so for all $n \geq 1$ and for all $K \geq 1$ we must have

$$\sum_{j=1}^{K} |x_j^{(n)}|^2 \leq C^2 \,. \tag{1}$$

Since (1) holds for all $n \geq 1$, taking the limit as $n \to \infty$, and, very importantly, minding the fact that we have a *finite* sum (only up to K, not to ∞), we obtain the following:

$$\sum_{j=1}^{K} |z_j|^2 \leq C^2 \,.$$

Now this holds for all $K \geq 1$, so taking the limit as $K \to \infty$ we conclude that

$$\sum_{j=1}^{\infty} |z_j|^2 \leq C^2 \,,$$

or in other words that $\|z\|_2^2 \leq C^2$, which shows that $\|z\|_2 \leq C$. Therefore, $z \in \ell^2(\mathbb{R})$.

Step 3: We prove that $z = \lim_{n\to\infty} x^{(n)}$ with respect to the $\|\cdot\|_2$ norm. We have to show that

$$\|x^{(n)} - z\|_2 \to 0 \quad \text{as } \; n \to \infty \,.$$

It is of course enough to show that

$$\sum_{j=1}^{\infty} |x_j^{(n)} - z_j|^2 = \|x^{(n)} - z\|_2^2 \to 0 \quad \text{as } \; n \to \infty \,.$$

Let $\epsilon > 0$. Since $\left(x^{(n)}\right)_{n\geq 1}$ is Cauchy with respect to the $\|\cdot\|_2$ norm, there is $N \in \mathbb{N}$ such that

$$\|x^{(n)} - x^{(m)}\|_2 \leq \epsilon \quad \text{for all } \ n, m \geq N\,,$$

so

$$\sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^2 = \|x^{(n)} - x^{(m)}\|_2^2 \leq \epsilon^2 \quad \text{for all } \ n, m \geq N\,.$$

For *every* integer $K \geq 1$, from the above we have

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^2 \leq \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^2 \leq \epsilon^2 \quad \text{for all } \ n, m \geq N\,.$$

Fix $K \geq 1$ and $n \geq N$. Let $m \to \infty$, so that $x_j^{(m)} \to z_j$. Thus we get

$$\sum_{j=1}^{K} |x_j^{(n)} - z_j|^2 \leq \epsilon^2.$$

This holds for all $K \geq 1$, so if we let $K \to \infty$ we get

$$\sum_{j=1}^{\infty} |x_j^{(n)} - z_j|^2 \leq \epsilon^2,$$

which proves that $\|x^{(n)} - z\|_2^2 \to 0$ and concludes our proof.

$\boxed{3}$ Let $(X, \|.\|)$ be a normed space.

- **a)** Show that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is bounded in X.
- **b)** Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $(y_n)_{n\in\mathbb{N}}$ another sequence in X. If we have that
$$\|y_n - y_m\| \leq \|x_n - x_m\|$$
for all $m, n \in \mathbb{N}$, then $(y_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in X.

Solution. **a)** Since (x_n) is a Cauchy sequence, by choosing say $\epsilon = 1$, there is $N \in \mathbb{N}$ such that

$$\|x_n - x_m\| \leq 1 \quad \text{for all } \ n, m \geq N\,.$$

In particular, putting $m = N$ we have

$$\|x_n - x_N\| \leq 1 \quad \text{for all } \ n \geq N\,.$$

Using the triangle inequality, for all $n \geq N$ we then have

$$\|x_n\| = \|x_n - x_N + x_N\| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\|\,.$$

Therefore, all but the first $N-1$ terms of the sequence (x_n) have norms bounded by the constant $1+\|x_N\|$. Since any finite set of real numbers has a maximum, we conclude that the whole sequence is bounded. More precisely, for all $n \geq 1$ we have

$$\|x_n\| \leq \max\left\{1+\|x_N\|, \|x_1\|, \ldots, \|x_{N-1}\|\right\}.$$

b) To show that (y_n) is Cauchy, we need to show that for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|y_n - y_m\| < \epsilon$$

whenever $m, n \geq N$. We therefore fix $\epsilon > 0$, and look for $N \in \mathbb{N}$ such that the inequality above holds. Since (x_n) is a Cauchy sequence, there exists $N_1 \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon$$

whenever $m, n \geq N_1$. However, we also have that $\|y_n - y_m\| \leq \|x_n - x_m\|$. Hence, for $m, n \geq N_1$ we have that

$$\|y_n - y_m\| \leq \|x_n - x_m\| < \epsilon$$

, so we can pick $N = N_1$ to show that (y_n) is Cauchy.

$\boxed{4}$ Prove the following two statements for a normed space $(X, \|.\|)$.

a) Any ball $B_r(x) = \{y \in X : \|x-y\| < r\}$ in $(X, \|.\|)$ is bounded and $\operatorname{diam}(B_r(x)) \leq 2r$.

b) If A is a bounded subset of $(X, \|.\|)$, then for any $a \in A$ we have $A \subseteq \bar{B}_{\operatorname{diam}(A)}(a)$. (Recall that the a closed ball $\bar{B}_r(x)$ is the set $\{y \in X : \|y-x\| \leq r\}$.)

Solution. Note that there are many ways of solving this, in particular because lemma 4.3 gives us a lot of equivalent statements to describe boundedness.
a) By part 2 of lemma 4.3, a set A is bounded if there is some constant $0 \leq M < \infty$ such that $\|x-y\| < M$ for every $x, y \in A$. This implies that $B_r(x)$ is bounded: if $y, z \in B_r(x)$, then

$$\|y-z\| = \|(y-x)+(x-z)\| \leq \|y-x\| + \|z-x\| \leq r + r = 2r,$$

where we have used the triangle inequality and the definition of $B_r(x)$. Hence we can pick $M = 2r$, and $B_r(x)$ is bounded. The diameter $\operatorname{diam}(B_r(x))$ is defined to be $\sup\{\|y-z\| : y, z \in B_r(x)\}$. Above we showed that $\|y-z\| \leq 2r$ for any $y, z \in B_r(x)$, hence $\operatorname{diam}(B_r(x)) = \sup\{\|y-z\| : y, z \in B_r(x)\} \leq 2r$.
b) Recall that $\operatorname{diam}(A)$ is defined to be $\sup\{\|y-z\| : y, z \in A\}$, and by part 3 of lemma 4.3 we know that $\operatorname{diam}(A) < \infty$. Now pick $a \in A$, and consider some $x \in A$. By the definition of the diameter, we have

$$\|a-x\| \leq \operatorname{diam}(A).$$

Since x was arbitrary, this shows exactly that $A \subseteq \bar{B}_{\operatorname{diam}(A)}(a)$, since $\bar{B}_{\operatorname{diam}(A)}(a) = \{x \in X : \|x-a\| \leq \operatorname{diam}(A)\}$.

$\boxed{5}$ **a)** Let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(t) = \begin{cases} 0 & \text{for } a \le t \le \frac{a+b}{2}, \\ n(t - \frac{a+b}{2}) & \text{for } \frac{a+b}{2} < t \le \frac{a+b}{2} + \frac{1}{n}, \\ 1 & \text{for } \frac{a+b}{2} + \frac{1}{n} \le t \le b. \end{cases}$$

in $C[a,b]$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$.

b) Let $(f_n)_{n\in\mathbb{N}}$ be the sequence on $[0,1]$ defined by $f_n(x) = \frac{1}{1+nx}$. Determine if $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,1]$.

Solution. a) As on the previous problem set, we highly recommend that you sketch some of these functions! It is not hard to see that f_n converge pointwise to the function f given by

$$f(t) = \begin{cases} 0 & \text{for } a \le t \le \frac{a+b}{2}, \\ 1 & \text{for } \frac{a+b}{2} < t \le b. \end{cases}$$

However, this convergence is not uniform. [1] Pick $\epsilon = \frac{1}{4}$. I claim that there is no $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \ge N$. Since each f_n is a continuous function with $f(a) = 0$ and $f(b) = 1$, there must for each n be some $x_{1/2} \in [a,b]$ with $f_n(x_{1/2}) = \frac{1}{2}$. However, the only values of the function f are 0 and 1, so we must have $|f(x_{1/2}) - f_n(x_{1/2})| = \frac{1}{2} \not< \frac{1}{4}$. Since we could find such a point $x_{1/2}$ for any n, it is clearly not possibly to find $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \ge N$.

b) Once again, a simple sketch is highly recommended. Clearly, for each fixed $x \in (0,1]$, we have that $\lim_{n\to\infty} f_n(x) = 0$, and $\lim_{n\to\infty} f_n(0) = 1$. hence f_n converges pointwise to

$$f(t) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } 0 < x \le 1. \end{cases}$$

To show that the convergence is not uniform, we can use more or less the same argument as above. Each f_n is a continuous function with $f_n(0) = 1$ and $f_n(1) = \frac{1}{1+n} \le \frac{1}{2}$, so for any n there must exist $x_{1/2} \in [a,b]$ with $f_n(x_{1/2}) = \frac{1}{2}$. As before the limit function f only takes the values 0 and 1, so by picking $\epsilon = \frac{1}{4}$ we will not be able to find $N \in \mathbb{N}$ such that

$$|f(x) - f_n(x)| < \frac{1}{4}$$

for any $x \in X$ whenever $n \ge N$.

[1] Since each f_n is a continuous function, we know from proposition 4.1.13 that any uniform limit of the f_n would be continuous. But clearly f is not continuous - this would be a slick proof. But the students should do this using ϵ-arguments.

$\boxed{6}$ Let f be a Lipschitz function $f : (X, \|.\|_X) \to (Y, \|.\|_Y)$. Show that f is continuous.

Solution. By definition of Lipschitz, there exists some constant L such that

$$\|f(x) - f(x')\| \leq L\|x - x'\|$$

for any $x, x' \in X$. To show that f is continuous we need to show that: for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(x')\| < \epsilon$ whenever $\|x - x'\| < \delta$. Therefore we fix some arbitrary $\epsilon > 0$ and look for δ. It is not too difficult to see that $\delta = \frac{\epsilon}{L}$ will work, because if $\|x - x'\| < \frac{\epsilon}{L}$, then

$$\|f(x) - f(x')\| \leq L\|x - y\| < L\frac{\epsilon}{L} = \epsilon.$$