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Exercise set 7: Solutions

Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Suppose A is a closed subspace of a Banach space $(X, \|.\|)$. Then $(A, \|.\|)$ is a complete subspace of X, i.e. $(A, \|.\|)$ is a Banach space.

Solution. By the definition of being closed, A contains all its limit points. This means that if $a_n \to x$ for some sequence $(a_n) \in A$, then $x \in A$. To show that A is complete, we assume that (a_n) is a Cauchy sequence in A. Since A is a closed subspace of X, (a_n) is also a Cauchy sequence in X. X is assumed to be complete, hence (a_n) converges to some $x \in X$. As A is closed, $x \in A$, which means that (a_n) converges in A to a.

2 Let $(X, \|.\|)$ be a normed spaces and $A \subseteq X$. Then

$$\overline{A} = \cap_{n\in\mathbb{N}}(A + B_{1/n}(0)),$$

where $A + B_{1/n}(0) = \{x \in X : x = a + y \quad a \in A,\, y \in B_{1/n}(0)\}$.

Solution. To show that two sets are equal, we must show that each set is a subset of the other. Assume that $a \in \overline{A}$ – we need to show that $a \in A + B_{1/n}(0)$ for every $n \in \mathbb{N}$. Since $a \in \overline{A}$, there is a sequence (a_n) in A converging to a. Therefore we can, for every $n \in \mathbb{N}$, pick some a_N such that $\|a - a_N\| < \frac{1}{n}$. In other words $a - a_N \in B_{1/n}(0)$, hence $a \in A + B_{1/n}(0)$.

Conversely, assume that $x \in A + B_{1/n}(0)$ for every $n \in N$. We can then define a sequence a_n in a converging to x as follows. By the assumption, for each $n \in \mathbb{N}$ there is some $a_n \in A$ and $y_n \in B_{1/n}(0)$ such that $x = a_n + y_n$ – this way we define the sequence (a_n). It is clear that $a_n \to x$, since $\|x - a_n\| = \|y_n\| < \frac{1}{n}$.

3 Suppose $(X, \|.\|)$ is a normed space.

a) Show that $B_r(x) = \{y \in X : \|x - y\| < r\}$ is an open set in X.

b) Show that singletons are closed sets, i.e. for any $x \in X$ we have that $\{x\}$ is closed.

Solution. **a)** We need to show that for any $y \in B_r(x)$ there is some $r_y > 0$ such that $B_{r_y}(y) \subset B_r(x)$. We may pick $r_y = r - \|y - x\| > 0$. If $z \in B_{r_y}(y)$, we find by the triangle inequality that

$$\|z - x\| \leq \|z - y\| + \|y - x\| < r_y + (r - r_y) = r,$$

hence $z \in B_r(x)$. This shows that $B_{r_y}(y) \subset B_r(x)$.
b) We need to show that $\overline{\{x\}} = \{x\}$, i.e. x is the only limit point of $\{x\}$. Assume that $y \in \overline{\{x\}}$. Then there is a sequence in $\{x\}$ converging to y. Of course, the only sequence in $\{x\}$ is the constant sequence $x_n = x$ for every x. Clearly this constant sequence converges to x, and since the limit of a sequence is unique we must conclude $y = x$, hence x is the only limit point of $\{x\}$.

$\boxed{4}$ Consider the integral equation

$$f(x) = \sin x + \lambda \int_0^3 e^{-(x-y)} f(y) dy$$

for some scalar λ.

a) Determine for which λ there exists a continuous function f on $[0, 3]$ that solves this integral equation.

b) Pick one of the values of λ found in **a)**. Use the method of iteration, as described in Banach's fixed point theorem, to find approximations f_1 and f_2 to a potential solution by starting with $f_0(x) = 1$ on $[0, 3]$.

Solution. **a)** We will use proposition 3.3.5 in the lecture notes. Some of the conditions for this proposition will always be satisfied:

- $\sin(x)$ is continuous on $[0, 3]$ (this is g in the notation of 3.3.5)
- $e^{-(x-y)}$ is continuous on $[0, 3] \times [0, 3]$ (k in 3.3.5).

But we also need that $|\lambda| < \frac{1}{3\|e^{-(x-y)}\|_\infty}$ for the solution to exist, by the same proposition. We easily find that

$$\|e^{-(x-y)}\|_\infty = \sup_{x,y \in [0,3]} |e^{-(x-y)}| = e^3.$$

(This is easy to see, since $e^{-(x-y)} = e^{-x}e^y$, so we just need to find the supremum of each factor and multiply them.). We conclude that a solution exists for $|\lambda| < \frac{1}{3e^3} \approx 0.017$.
b) We pick $\lambda = 1/100$. With $f_0(x) = 1$ for $x \in [0, 3]$, we find by iteration that

$$f_1(x) = \sin(x) + 0.01 \int_0^3 e^{-(x-y)}\ dy = \sin(x) + 0.01(e^3 - 1)e^{-x}.$$

We insert this back into the iteration once again, to obtain

$$\begin{aligned} f_2(x) &= \sin(x) + 0.01 \int_0^3 e^{-(x-y)} \left(\sin(y) + 0.01(e^3 - 1)e^{-y}\right) \, dy \\ &= \sin(x) + 0.01 \left(0.01(e^3 + e^{-3} - 2) + \frac{1}{2} e^{-x}(1 + e^3 \sin(3) - e^3 \cos(3))\right) \\ &= C + \sin(x) + Ke^{-x} \end{aligned}$$

where C, K are constants that you may calculate. It is not difficult to see that the function f_n after n iterations will also be of the same form, namely

$$f_n(x) = C_n + sin(x) + K_n e^{-x}.$$

$\boxed{5}$ Let A be a non-empty subset of a normed space $(X, \|.\|)$.

a) Show that the closure of the linear span of A is a closed subspace of X, denoted by $\overline{\text{span}(A)}$.

b) We define the *closed linear span* of A, denoted by $\overline{span}(A)$, as the intersection of all the closed linear subspaces containing A. Show that $\overline{span}(A) = \overline{\text{span}(A)}$.

Solution. a) By lemma 4.4, the closure of a subset is always closed. Hence we only need to show that $\overline{\text{span}(A)}$ is a subspace – i.e. closed under addition and scalar multiplication. Assume that λ is a scalar and that $x, y \in \overline{\text{span}(A)}$. Since the closure is the set of limit points, this means that there exist sequences (x_n) and (y_n) in $\text{span}(A)$ such that

$$\begin{aligned} \lim_{n\to\infty} x_n &= x \\ \lim_{n\to\infty} y_n &= y. \end{aligned}$$

We want to show that $x + \lambda y \in \overline{\text{span}(A)}$. We clearly have that

$$\lim_{n\to\infty} x_n + \lambda y_n = x + \lambda y_n.$$

Hence $x + \lambda y$ is the limit of the sequence $(x_n + \lambda y_n)$, and since $\text{span}(A)$ is a subspace we know that $(x_n + \lambda y_n)$ is a sequence in $\overline{\text{span}(A)}$ [1]. Since the closure is the set of all limit points, this means that $x + \lambda y \in \overline{\text{span}(A)}$.

b) By lemma 4.6 we know that the closure $\overline{\text{span}(A)}$ is the intersection of all closed *sets* containing $\text{span}(A)$. This immediately gives us that $\overline{\text{span}(A)} \subset \overline{span}(A)$, since $\overline{span}(A)$ is the intersection of all *subspaces* of $\text{span}(A)$.

On the other hand, we have by **a)** that $\overline{\text{span}(A)}$ is a closed subspace containing $\text{span}(A)$. Hence $\overline{span}(A) \subset \overline{\text{span}(A)}$, since the set on the left is the intersection of all closed subspaces containing $\text{span}(A)$.

[1] What we mean by this is simply that $x_n + \lambda y_n \in \text{span}(A)$ for every $n \in \mathbb{N}$.