Exercise set 8: Solutions

Note: When showing that $\|T\| = C$ for an operator T and a positive number C, we must show that $C = \sup\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$. One often divides the proof into two steps:

1. Show that $\|T\| \leq C$, often using a simple calculation. This means that C is some upper bound of $\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$.

2. Show that $\|T\| = C$ by showing that C is the *least* upper bound of $\{\frac{\|Tx\|}{\|x\|} : x \neq 0\}$.

These two steps will be used in problems 3,4 and 5.

1 Let X and Y be normed spaces.

a) Show that $f : X \to Y$ is continuous if and only if for any closed set $F \subset Y$ its preimage $f^{(-1)}(F)$ is closed in X.

b) Show that the zero set $\{x \in X : f(x) = 0\}$ of a continuous function $f : X \to Y$ is closed. Use the preceding statement.

c) Use the preceding statement to prove that the kernel of a bounded linear transformation $T : X \to Y$ is a closed subspace.

Solution. a) Let us first quickly prove the hint.

$$\begin{aligned} f^{(-1)}(Y \setminus A) &= \{x \in X : f(x) \in Y \setminus A\} \\ &= \{x \in X : f(x) \notin A\} \\ &= X \setminus \{x \in X : f(x) \in A\} \\ &= X \setminus f^{(-1)}(A). \end{aligned}$$

Now recall 2 facts:

1. A function is continuous if and only if the preimage of any open set is open

2. A set is closed if and only if its complement is open.

Now assume that f is continuous and $F \subset Y$ is closed. Then $Y \setminus F$ is open, and by the continuity of f we have that $f^{(-1)}(Y \setminus F) = X \setminus f^{(-1)}(F)$ is open in X. By this means that the complement of $f^{(-1)}(F)$ is open – hence $f^{(-1)}(F)$ is closed in X.
Conversely, assume that $f^{(-1)}(F)$ is closed in X for any closed $F \subset Y$. Then let $U \subset Y$ be open – we need to show that $f^{(-1)}(U)$ is open in X to show that f is continuous. Since U is open, its complement $Y \setminus U$ is closed. By assumption we then get that $f^{(-1)}(Y \setminus U) = X \setminus f^{(-1)}(U)$ is closed in X. This means that the complement of $f^{(-1)}(U)$ is closed, hence $f^{(-1)}(U)$ is open.
b) By a problem on the previous problem set, the set $\{0\}$ is closed. By part a) this means that $f^{(-1)}(\{0\}) = \{x \in X : f(x) = 0\}$ is closed.
c) From the lecture notes we know that a linear transformation between normed spaces is continuous if and only if it is bounded. Hence if T is bounded it is continuous, and so $\ker(T) = \{x \in X : T(x) = 0\}$ is closed by part b).

$\boxed{2}$ Let X and Y be normed spaces. Show that a linear map $T : X \to Y$ is not continuous if and only if there exists a sequence of unit vectors (x_n) in X such that $\|Tx_n\| \geq n$ for $n \in \mathbb{N}$.

Solution. Since T is a linear mapping between normed spaces, it is bounded if and only if it is continuous. Assume that T is not bounded. Then by lemma 4.12 (3) we have that

$$\sup\{\|Tx\|_Y : \|x\|_X = 1\} = \infty.$$

Hence no $n \in \mathbb{N}$ is an upper bound for the set $\{\|Tx\|_Y : \|x\|_X = 1\}$, which means that for every $n \in \mathbb{N}$ there exists some $x_n \in X$ with $\|x_n\|_X = 1$ and $\|Tx_n\|_Y > n$. This defines a sequence (x_n) with the desired properties.
Conversely, assume that we have a sequence (x_n) of unit vectors with $\|Tx_n\| \geq n$ for every $n \in \mathbb{N}$. Since the supremum is an upper bound, we must then have

$$\sup\{\|Tx\|_Y : \|x\|_X = 1\} \geq n \text{ for every } n \in \mathbb{N}.$$

Hence $\sup\{\|Tx\|_Y : \|x\|_X = 1\} = \infty$, so the operator is not bounded.

$\boxed{3}$ Let T be a linear mapping $T : (\mathbb{R}^n, \|.\|_\infty) \to (\mathbb{R}^n, \|.\|_\infty)$ given by a $n \times n$ matrix A. Show that the operator norm of T in terms of A is given by $\|T\| = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$.

Solution.
Please note that there is an example at the end of the solution, to illustrate what we are doing. Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Then the vector $Ax = ((Ax)_1, (Ax)_2, ...(Ax)_n)$ is given by

$$(Ax)_i = \sum_{j=1}^n a_{ij} x_j,$$

by the definition of the product of a matrix and a vector. Hence

$$\|Ax\|_\infty = \max_{i=1,\ldots,n} |\sum_{j=1}^n a_{ij}x_j| \leq \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}||x_j| \leq \|x\|_\infty \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|,$$

where we have used the triangle inequality and the fact that $|x_i| \leq \|x\|_\infty$ for any $1 \leq i \leq n$. This shows that

$$\frac{\|Tx\|_\infty}{\|x\|_\infty} \leq \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}| \quad \text{for } x \neq 0, \tag{1}$$

and hence $\|T\| \leq \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|$ since $\|T\| = \sup\{\frac{\|Tx\|_\infty}{\|x\|_\infty} : x \neq 0\}$.
One way to show that $\|T\| = \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|$ would be to find some vector $x \in \mathbb{R}^n$ such that

$$\frac{\|Tx\|_\infty}{\|x\|_\infty} \geq \max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|. \tag{2}$$

This would show that $\max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}|$ is not only an upper bound for $\{\frac{\|Tx\|_\infty}{\|x\|_\infty} : x \neq 0\}$, as we saw in equation 1, but actually the *least* upper bound. Let us therefore look for an x satisfying (2).
Assume that the maximal row sum is achieved for the index k – meaning that $\max_{i=1,\ldots,n} \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{kj}|$. We are looking for $x \in \mathbb{R}^n$ such that

$$\frac{\|Tx\|_\infty}{\|x\|_\infty} \geq \sum_{j=1}^n |a_{kj}|.$$

Define $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ by

$$x_j = \begin{cases} 1 & \text{if } a_{kj} > 0 \\ 0 & \text{if } a_{kj} = 0 \\ -1 & \text{if } a_{kj} < 0. \end{cases}$$

In other words, we construct x from the row $(a_{k1}, a_{k2}, ..., a_{kn})$ of A by defining the j'th element of x to be the *sign* of the j'th element of this row. This means that $x_j a_{kj} = |a_{kj}|$ for every $j = 1, 2, ...n$. We then get that

$$(Tx)_k = (Ax)_k = \sum_{j=1}^n a_{kj}x_j = \sum_{j=1}^n |a_{kj}|,$$

where the last equality follows from the way we defined x. This means that $\|Tx\|_\infty = \sum_{j=1}^n |a_{kj}|$ by the definition of the ∞-norm. Clearly $\|x\|_\infty = 1$ [1]. Hence

$$\frac{\|Tx\|_\infty}{\|x\|_\infty} \geq \sum_{j=1}^n |a_{kj}|,$$

which is what we needed to show.

[1] We could in theory also have that $\|x\|_\infty = 0$, but this means that each $a_{kj} = 0$, and hence $\|T\| = 0$ since $\|T\| = \sum_{j=1}^n |a_{kj}|$.

Example *By no means expected as part of a solution – for pedagogical reasons only.* Let A be defined by

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & -5. \end{pmatrix}$$

In this case $n = 3$, and we would like to understand what the expression $\max_{i=1,2,3} \sum_{j=1}^{3} |a_{ij}|$ means in this case. Note that i denotes the row number, so for a fixed i the sum $\sum_{j=1}^{3} |a_{ij}|$ is the sum of the absolute value of row i. In our case:

- For $i = 1$, the sum is $2 + |-3| + 1 = 6$.
- For $i = 2$, the sum is $0 + 2 + 2 = 4$.
- For $i = 3$, the sum is $3 + 0 + |-5| = 8$.

The expression $\max_{i=1,2,3} \sum_{j=1}^{3} |a_{ij}|$ is the maximum of these three numbers, and clearly the maximum is achieved in the third row and equals 8. The second part of the solution constructed some vector x such that $\|Ax\|_\infty \geq \max_{i=1,2,3} \sum_{j=1}^{3} |a_{ij}|$. In our case we know that the last expression actually equals 8, and is the sum of the absolute values of the elements in the third row. How can we then find some unit vector $x \in \mathbb{R}^3$ such that $\|Ax\|_\infty \geq 8$? We can do this by noting that element 3 in the vector Ax is the dot product between the third row of A and x:

$$Ax = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ 3x_1 + 0x_2 - 5x_3 \end{pmatrix}$$

So in order to make the third element of Ax be equal to 8, we could obviously choose $x = (1, 0, -1)$. Then $\|Ax\|_\infty \geq 8$, and it was such a vector x that we needed to construct. This is the reasoning used to construct the solution in the general case.

$\boxed{4}$ Let T be the integral operator $Tf(x) = \int_0^1 k(x,y)f(y)dy$ defined by a kernel $k \in C([0,1] \times [0,1])$ such that $k(x,y) \geq 0$ for any $(x,y) \in [0,1] \times [0,1]$. Show that the operator norm of T as a mapping on $C[0,1]$ with respect to $\|.\|_\infty$-norm is $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$.

Solution. Let $f \in C[0,1]$. We calculate using the triangle inequality for integrals that

$$\begin{aligned}
\|Tf\|_\infty &= \max_{x\in[0,1]} \left| \int_0^1 k(x,y)f(y)dy \right| \\
&\leq \max_{x\in[0,1]} \int_0^1 |k(x,y)||f(y)|dy \\
&\leq \|f\|_\infty \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy \\
&= \|f\|_\infty \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy.
\end{aligned}$$

As we did in problem 3, we may conclude from this inequality that $\|T\| \leq \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$. We would now like to show that $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$, and similarly to problem (3) it will be enough to find some function $f \in C[0,1]$ such that $\frac{\|Tf\|_\infty}{\|f\|_\infty} \geq \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$. Assume that the integral $\int_0^1 |k(x,y)|dy$ attains its maximum at the point x'– meaning that $\max_{x\in[0,1]} \int_0^1 |k(x,y)|dy = \int_0^1 |k(x',y)|dy$. If we pick f to be the constant function $f(x) = 1$, then

$$Tf(x') = \int_0^1 k(x',y)dy = \int_0^1 |k(x',y)|dy,$$

which implies that $\|Tf\|_\infty \geq \int_0^1 |k(x',y)|f(y)dy$, and hence

$$\frac{\|Tf\|_\infty}{\|f\|_\infty} \geq \int_0^1 |k(x',y)|dy = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy,$$

which is what we needed to show.
Note: We have used that k is positive to get that $|k(x,y)| = k(x,y)$ for $x, y \in \mathbb{R}$. If this was not true, we would not get that $\|Tf\|_\infty = \int_0^1 |k(x',y)|f(y)dy$ when f is the constant function 1. The obvious way of fixing this would be to follow in the steps of problem (3), and replace this constant function with f defined by

$$f(y) = \begin{cases} 1 & \text{if } k(x',y) > 0 \\ 0 & \text{if } k(x',y) = 0 \\ -1 & \text{if } k(x',y) < 0. \end{cases}$$

The problem with this procedure is that this function is not continuous in general, which is a big problem since we are working in the space of continuous functions! Of course, if k is a positive function, this f will simply be the constant function 1. There are two ways of getting rid of this problem:

1. By a slightly more technical proof, it is possible to show that $\|T\| = \max_{x\in[0,1]} \int_0^1 |k(x,y)|dy$ even when k is not positive.

2. One could also change the setting of the problem by considering T to be an operator from $L^\infty([0,1])$ to $L^\infty([0,1])$. Here $L^\infty([0,1])$ is the much larger space of all bounded functions on $[0,1]$ – they don't have to be continuous. You will meet this space later in the course. In particular, the f that we defined

by taking the sign of $k(x', y)$ actually belongs to $L^\infty([0,1])$, since its absolute value never exceeds 1. Hence we may apply T to this f, and we would find that

$$\|Tf\|_\infty = \int_0^1 |k(x', y)dy.$$

$\boxed{5}$ Let T be a linear operator from $(\ell^\infty, \|.\|_\infty)$ to $(\ell^\infty, \|.\|_\infty)$ defined by an infinite matrix $(a_{ij})_{i,j=1}^\infty$ satisfying $\sum_{j=1}^\infty |a_{ij}| < \infty$. Show that the operator norm of T is given by $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$.

Solution. Let $x = (x_1, x_2, ...) \in \ell^\infty$. Then the vector Tx is given by

$$(Tx)_i = \sum_{j=1}^\infty a_{ij}x_j,$$

by the definition of the product of a matrix and a vector. Hence we find using the triangle inequality and the fact that $|x_i| \le \|x\|_\infty$ for any $i \in \mathbb{N}$ that

$$\|Tx\|_\infty = \sup_{i\in\mathbb{N}} |\sum_{j=1}^\infty a_{ij}x_j| \le \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}||x_j| \le \|x\|_\infty \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|.$$

This shows that

$$\frac{\|Tx\|_\infty}{\|x\|_\infty} \le \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| \quad \text{for } x \neq 0,$$

and hence $\|T\| \le \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$.
It remains to show that $\|T\| = \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$. In the previous two problems we used that we had a maximum, but now we cannot guarantee that there is a maximum – we need to adapt our tactics slightly. By definition

$$\|T\| = \sup\{\frac{\|Tx\|_\infty}{\|x\|_\infty} : x \neq 0\},$$

and we have shown that $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$ is an upper bound for $\{\frac{\|Tx\|_\infty}{\|x\|_\infty} : x \neq 0\}$. To show that $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$ is the least upper bound, we need for every $\epsilon > 0$ to find some $x^\epsilon \in \ell^\infty$ such that[2]

$$\frac{\|Tx^\epsilon\|_\infty}{\|x^\epsilon\|_\infty} > \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| - \epsilon.$$

We therefore fix $\epsilon > 0$, and look for such an x^ϵ. There is some index k_ϵ such that

$$\sum_{j=1}^\infty |a_{k_\epsilon j}| > \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| - \epsilon.$$

[2]Why do we need this? This will show that $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| - \epsilon$ is *not* an upper bound for $\{\frac{\|Tx\|_\infty}{\|x\|_\infty} : x \neq 0\}$ for any $\epsilon > 0$. Clearly we must then conclude that $\sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}|$ is the *least* upper bound!

This is a consequence of the definition of the supremum as the least upper bound. We now follow what we did in problem (3), and define a sequence $x^\epsilon \in \ell^\infty$ by

$$x_j^\epsilon = \begin{cases} 1 & \text{if } a_{k_\epsilon j} > 0 \\ 0 & \text{if } a_{k_\epsilon j} = 0 \\ -1 & \text{if } a_{k_\epsilon j} < 0. \end{cases}$$

This means that $x_j^\epsilon a_{k_\epsilon j} = |a_{k_\epsilon j}|$ for every $j \in \mathbb{N}$. Clearly $\|x^\epsilon\|_\infty = 1$ and element k_ϵ of the sequence Tx^ϵ is given by

$$\begin{aligned}(Tx^\epsilon)_{k_\epsilon} &= \sum_{j=1}^\infty a_{k_\epsilon j} x_j^\epsilon \\ &= \sum_{j=1}^\infty |a_{k_\epsilon j}| > \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| - \epsilon.\end{aligned}$$

In particular, this means that $\|Tx^\epsilon\|_\infty \geq \sup_{i\in\mathbb{N}} \sum_{j=1}^\infty |a_{ij}| - \epsilon$. But this shows that x^ϵ is exactly the kind of element we needed to construct.

$\boxed{6}$ Let T be a linear operator between the normed spaces X and Y. We say that T is an isometry if $\|Tx\|_Y = \|x\|_X$ for all $x \in X$.

a) Show that if T is an isometry, then T is injective.

b) For $1 \leq p \leq \infty$ define the shift operator $T : \ell^p \to \ell^p$ by $Tx = (0, x_1, x_2, ...)$. Show that T is an isometry and determine its range and kernel.

Solution: a) Assume that $Tx = Tx'$ for some $x, x' \in X$. By the linearity of T, this implies that $T(x - x') = 0$. Since T is an isometry, we get that $\|0\| = \|T(x - x')\| = \|x - x'\|$. By the positivity axiom for normed spaces, this implies that $x = x'$, hence T is injective.

b) Let $y = Tx$ for $x \in X$. Then $y = (0, x_1, x_2, ...)$, which we can express by saying that $y_1 = 0$ and $y_n = x_{n-1}$ for $n > 1$. For $p < \infty$, we find that

$$\begin{aligned}\|T_x\|_p = \|y\|_p &= \left(\sum_{n=1}^\infty |y_n|^p\right)^{1/p} \\ &= \left(0 + \sum_{n=2}^\infty |y_n|^p\right)^{1/p} \\ &= \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \\ &= \|x\|_p.\end{aligned}$$

For $p = \infty$ we find that

$$\begin{aligned}\|T_x\|_\infty = \|y\|_\infty &= \sup_{n\in\mathbb{N}} |y_n| \\ &= \sup_{n\in\mathbb{N}} |x_n| \\ &= \|x\|_\infty,\end{aligned}$$

where we have used that the *sets* $\{y_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ are equal, except for the fact that $y_1 = 0$. Clearly the kernel of T is just the zero sequence, and the range consists of all sequences in ℓ^p such that the first element is 0.