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Please justify your answers! The most important part is *how* you arrive at an answer, not the answer itself.

1 Use the Banach fixed point theorem to solve:

$$\begin{aligned} 7x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + 3x_2 + x_3 &= 2 \\ x_1 - x_2 + 5x_3 &= 1 \end{aligned}$$

Hint: Pick appropriate norms on $\mathbb{R}^3$ to get a contraction.

Solution. As you have seen in the lectures, we will formulate the problem as a fixed-point problem of the form $x = Ax + b$. The system of equations is equivalent to

$$\begin{aligned} x_1 &= \frac{1}{7} + \frac{1}{7}x_2 - \frac{2}{7}x_3 \\ x_2 &= \frac{2}{3} + \frac{1}{3}x_1 - \frac{1}{3}x_3 \\ x_3 &= \frac{1}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_2 \end{aligned}$$

which we may write in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/7 & -2/7 \\ 1/3 & 0 & -1/3 \\ -1/5 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/7 \\ 2/3 \\ 1/5 \end{bmatrix}$$

If we define

$$A = \begin{bmatrix} 0 & 1/7 & -2/7 \\ 1/3 & 0 & -1/3 \\ -1/5 & 1/5 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1/7 \\ 2/3 \\ 1/5 \end{bmatrix}$$

our problem becomes solving $x = Ax + b$ – a fixed point problem. In order to apply Banach's fixed point theorem, we need to have a contraction. In this case we need that

$$\|Ax + b - (Ay + b)\| = \|A(x - y)\| \leq K\|x - y\|$$

for any $x, y \in \mathbb{R}^3$ in some norm $\|.\|$ on $\mathbb{R}^3$. Let us use the $\|.\|_\infty$ on $\mathbb{R}^3$. From the last problem set, we then know that the operator norm of the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $Tx = Ax$ is the maximal row sum of the matrix A. In this case the maximal row sum appears in row 2 and equals $1/3 + 1/3 = 2/3$. Hence $\|T\| = 2/3$. But this means that

$$\|A(x-y)\|_\infty = \|T(x-y)\|_\infty \leq \|T\|\|x-y\|_\infty = \frac{2}{3}\|x-y\|_\infty.$$

Hence we have a contraction with $K = \frac{2}{3}$. By Banach's fixed point theorem we may choose any $x_0 \in \mathbb{R}^3$, and the iteration procedure $x_n = Ax_{n-1} + b$ will always converge to a solution x of $x = Ax + b$. Let us for instance pick

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the first few iterations give

$$x_1 = \begin{bmatrix} 0 \\ 2/3 \\ 1/5 \end{bmatrix}, ..., x_{10} = \begin{bmatrix} 0.1471 \\ 0.6176 \\ 0.2941 \end{bmatrix}$$

And you may check that this is a very good approximation to the solution of the original system.

2 We denote by c_f the set of all sequences with only finitely many non-zero entries.

a) For $1 \leq p < \infty$ show that c_f is dense in ℓ^p.

b) For $1 \leq p < \infty$ show that ℓ^p is separable.

Solution. **a)** Let $\epsilon > 0$ and $x \in \ell^p$ be given. We will show that we can find a sequence $y \in c_f$ such that $\|x - y\|_p < \epsilon$ – this would show that c_f is dense in ℓ^p. Since $x \in \ell^p$, we have by definition that $\sum_{i=1}^\infty |x_i|^p < \infty$. This means that there is some $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} |x_i|^p < \epsilon^p,$$

since the tail of a convergent series approaches zero. Define the finite sequence y by

$$y_i = \begin{cases} x_i & \text{for } 1 \leq i \leq N \\ 0 & \text{for } N+1 \leq i < \infty. \end{cases}$$

Then we find that

$$\begin{aligned}\|x-y\|_p &= \left(\sum_{i=1}^{\infty}|x_i-y_i|^p\right)^{1/p} \\ &= \left(\sum_{i=N+1}^{\infty}|x_i|^p\right)^{1/p} \\ &< (\epsilon^p)^{1/p} = \epsilon.\end{aligned}$$

b) To show that ℓ^p is separable, we need to find a countable, dense subset $A \subset \ell^p$. We will choose A to be those sequences in c_f with only rational elements. More precisely,

$$A = \{x \in \ell^p : x_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N} \text{ and } x_i \neq 0 \text{ for only finitely many } i \in \mathbb{N}\}.$$

We need to show that A is dense and countable. To show that A is countable, note that we can write A as the union

$$A = \bigcup_{n=1}^{\infty} A_n$$

where

$$A_n = \{x \in \ell^p : x_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N} \text{ and } x_i = 0 \text{ for all } i > n\}.$$

In words, A_n is the set of sequences $x \in \ell^p$ with rational coefficients and only the first n elements are allowed to be non-zero. We can identify A_n with $\mathbb{Q}^n$, where $\mathbb{Q}^n$ is the Cartesian product of $\mathbb{Q}$ with itself n times. After all, A_n consists of sequences with rational coefficients and only the first n elements are allowed to be non-zero. Such a sequence may clearly be identified with an n-tuple of rational numbers, i.e. an element of $\mathbb{Q}^n$. By proposition 1.3.7 the Cartesian product of countable sets is countable [1], hence $\mathbb{Q}^n$ is countable and therefore A_n is countable. Furthermore, A is the countable union of the countable sets A_n, hence A is countable by proposition 1.3.7.
Now we need to show that A is dense. Let $x \in \ell^p$ and $\epsilon > 0$ be given. We need to find $a \in A$ such that $\|x-a\|_p < \epsilon$. By (a) we may find $y \in c_f$ such that

$$\|x-y\|_p < \epsilon/2.$$

We would like to approximate y with some $a \in A$. Since $y \in c_f$, it has finitely many non-zero elements – assume that y has m non-zero elements. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we may for every $1 \leq i \leq m$ find some $q_i \in \mathbb{Q}$ such that

$$|y_i - q_i| < \frac{\epsilon}{2m^{1/p}}.$$

Now define the sequence $a \in A$ by

$$a_i = \begin{cases} q_i & \text{for } 1 \leq i \leq m \\ 0 & \text{for } m+1 \leq i < \infty. \end{cases}$$

[1] But we are only allowed to take *finite* products!

Then we find that

$$\begin{aligned}\|y-a\|_p &= \left(\sum_{i=1}^{\infty}|y_i-a_i|^p\right)^{1/p}\\ &= \left(\sum_{i=1}^{m}|y_i-r_i|^p\right)^{1/p}\\ &< \left(\sum_{i=1}^{m}\left(\frac{\epsilon}{2m^{1/p}}\right)^p\right)^{1/p}\\ &= \left(m\frac{1}{m}\left(\frac{\epsilon}{2}\right)^p\right)^{1/p}\\ &= \frac{\epsilon}{2}.\end{aligned}$$

Using the triangle inequality we then get

$$\|x-a\|_p \leq \|x-y\|_p + \|y-a\|_p < \epsilon/2 + \epsilon/2 = \epsilon.$$

Note. The main point of this exercise is to identify that A is the correct set to consider. You should then note that A is countable since $\mathbb{Q}$ is countable, and A is dense in ℓ^p since $\mathbb{Q}$ is dense in $\mathbb{R}$. As we have seen there are many details to this, but these are the main ideas.

3 Let M be a subspace of a Hilbert space X. Show that the orthogonal complement $M^{\perp} = \{x \in X : \langle x, y\rangle = 0 \text{ for all } y \in M\}$ is a subspace of X.

Solution. Clearly $0 \in M^{\perp}$. We need to show that $M^{\perp}$ is closed under addition and scalar multiplication. Assume that $x, x' \in M^{\perp}$. For every $y \in M$ we find that

$$\langle x + x', y\rangle = \langle x, y\rangle + \langle x', y\rangle = 0 + 0 = 0.$$

This show that $x + x' \in M^{\perp}$.
Then assume that $x \in M^{\perp}$ and λ is a scalar. For any $y \in M$ we find that

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle = \lambda \cdot 0 = 0.$$

Hence $\lambda x \in M^{\perp}$.

4 Consider the integral operator $T : (C[0,1], \|.\|_\infty) \to (C[0,1], \|.\|_\infty)$

$$Tf(x) = \int_0^1 k(x,y)f(y)dy,$$

where k is given by

$$k(x,y) = \sum_{i=1}^{n} g_i(x)h_i(y)$$

for $g_1, ..., g_n$ and $h_1, ..., h_n$ are continuous functions on $[0,1]$. We assume that $\{g_1, ..., g_n\}$ are linearly independent.

a) Determine the kernel and the range of T.

b) Investigate if the range of T is closed.

Solution. Let us start by rewriting the expression for T using the expression we have for k.

$$\begin{aligned} Tf(x) &= \int_0^1 k(x,y)f(y)dy \\ &= \int_0^1 \sum_{i=1}^n g_i(x)h_i(y)f(y)dy \\ &= \sum_{i=1}^n g_i(x)\int_0^1 h_i(y)f(y)dy. \end{aligned}$$

a) The kernel of T is the set of functions f such that $Tf = 0$. By our expression for T, this means that

$$\sum_{i=1}^n g_i(x)\int_0^1 h_i(y)f(y)dy = 0,$$

and since the functions g_i are assumed to be linearly independent this implies that

$$\int_0^1 h_i(y)f(y)dy = 0 \text{ for any } 1 \leq i \leq n.$$

So we have showed that

$$\ker(T) = \{f \in C[0,1] : \int_0^1 h_i(y)f(y)dy = 0 \text{ for any } 1 \leq i \leq n\}.$$

The range of T is the set of functions g such that

$$g(x) = \sum_{i=1}^n g_i(x)\int_0^1 h_i(y)f(y)dy$$

for some $f \in C[0,1]$. In particular g is a linear combination of the g_i.

b) By a), the range of T is a subspace of the finite-dimensional subspace spanned by the vectors $\{g_i : 1 \leq i \leq n\}$. Hence the range of T is a finite-dimensional subspace. The range of T is therefore closed, since any finite-dimensional subspace of a normed space is closed – let us prove this.

Assume that A is a subspace of a normed space X, and that A has the basis $\{e_i : 1 \leq i \leq n\}$. Then assume that (a_m) is a sequence in A that converges to some $x \in X$ in the norm of X, which we denote by $\|.\|$. To show that A is closed, we need to show that $x \in A$. We may define another norm $\|.\|_1$ on A by

$$\|\sum_{i=1}^n \lambda_i e_i\|_1 = \sum_{i=1}^n |\lambda_i|$$

for scalars $\{\lambda_i\}_{i=1}^n$. Each sequence element a_m can be written as a linear combination of the basis elements:

$$a_m = \sum_{j=1}^n c_j^{(m)} e_j$$

for scalars $\{c_j^{(m)}\}_{j=1}^n$. The sequence (a_m) is Cauchy in $\|.\|$, and since all norms on finite-dimensional spaces are equivalent (a_m) is Cauchy in $\|.\|_1$. The fact that (a_m) is Cauchy in $\|.\|_1$ implies, by the definition of the norm $\|.\|_1$, that the sequences $(c_j^{(m)})_{m=1}^\infty$ are Cauchy for each fixed j. Since $(c_j^{(m)})_{m=1}^\infty$ is a Cauchy sequence in the complete space $\mathbb{R}$ or $\mathbb{C}$, it converges to some c_j. Define

$$a = \sum_{j=1}^n c_j e_j.$$

It is simple to show that (a_j) converges to a in the norm $\|.\|_1$. But clearly $a \in A$, and since the norms $\|.\|_1$ and $\|.\|$ are equivalent we must have that (a_i) converges to a in the norm $\|.\|$. Since limits are unique we must conclude that $a = x$, which is what we needed to show.

5 Suppose $\|.\|_a$ and $\|.\|_b$ are equivalent norms on X. Then $(X, \|.\|_a)$ is a Banach space if and only if $(X, \|.\|_b)$ is a Banach space.

Solution. We will use lemma 4.15 of the notes. Assume that $(X, \|.\|_a)$ is a Banach space, and let (x_n) be a Cauchy sequence in the norm $\|.\|_b$. By part (2) of lemma 4.15, (x_n) is also Cauchy in the norm $\|.\|_a$. Since $(X, \|.\|_a)$ is a Banach space, this means that (x_n) converges in the norm $\|.\|_a$. By part (1) of lemma 4.15, we get that (x_n) converges in the norm $\|.\|_b$. This shows that $(X, \|.\|_b)$ is a Banach space.

The same proof will show that $(X, \|.\|_a)$ is a Banach space if $(X, \|.\|_b)$ is a Banach space.

6 Let $\|.\|_a$ and $\|.\|_b$ be two norms on a vector space X. Show that the following statements are equivalent:

1. $\|.\|_a$ and $\|.\|_b$ are equivalent norms.
2. For a set $U \subseteq X$ we have that U is open in $(X, \|.\|_a)$ if and only if U is open in $(X, \|.\|_b)$.

Solution. We start by assuming that $\|.\|_a$ and $\|.\|_b$ are equivalent norms. This means that we have constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_a \leq \|x\|_b \leq C_2\|x\|_a$$

for all $x \in X$. Then assume that U is open in $(X, \|.\|_a)$, and pick any $x \in U$. To show that U is open in $(X, \|.\|_b)$, we need to find some open ball $B_\epsilon^b(x)$ centered at x that such that $B_\epsilon^b(x) \subset U$. Since U is open in $(X, \|.\|_a)$, there exists some $r > 0$ such that $B_r^a(x) \subset U$. Since $B_r^a(x) \subset U$ and we want $B_\epsilon^b(x) \subset U$, it will clearly be enough to find $\epsilon > 0$ such that

$$B_\epsilon^b(x) \subset B_r^a(x).$$

The key to finding this ϵ is the inequality

$$\|x\|_a \leq \frac{1}{C_1}\|x\|_b.$$

I claim that if we pick $\epsilon = C_1 r$, then $B^b_\epsilon(x) \subset B^a_r(x) \subset U$.
To prove this, assume that $y \in B^b_\epsilon(x)$. We then find that

$$\begin{aligned}\|x-y\|_a &\leq \frac{1}{C_1}\|x-y\|_b \\ &< \frac{1}{C_1}\epsilon = r,\end{aligned}$$

hence $y \in B^a_r(x)$. The proof that any set U that is open in $(X, \|.\|_a)$ whenever U is open in $(X, \|.\|_b)$ is proved the same way, just using the inequality

$$\|x\|_b \leq C_2\|x\|_a.$$

Now assume that U is open in $(X, \|.\|_a)$ if and only if U is open in $(X, \|.\|_b)$. In the lecture notes we have proved that $\|.\|_a$ and $\|.\|_b$ are equivalent if and only if $B^a_{1/r}(0) \subset B^b_1(0) \subset B^a_r(0)$ for some $r > 0$ – hence it will be enough to find such an $r > 0$. We begin by considering $B^b_1(0)$. This is of course an open set in $(X, \|.\|_b)$, and by assumption it is therefore open in $(X, \|.\|_a)$. By our definition of open sets, this means that there must exist some $C_1 > 0$ such that $B^a_{C_1}(0) \subset B^b_1(0)$. By the same argument there must exist some $C_2 > 0$ such that $B^b_{C_2}(0) \subset B^a_1(0)$. This inclusion actually implies that $B^b_1(0) \subset B^a_{1/C_2}(0)$. Assuming this for now, we have shown that

$$B^a_{C_1}(0) \subset B^b_1(0) \subset B^a_{1/C_2}(0).$$

If we pick $r = \max\{\frac{1}{C_1}, \frac{1}{C_2}\}$, then this clearly implies that

$$B^a_{1/r}(0) \subset B^b_1(0) \subset B^a_r(0).$$

It only remains to justify the assertion that $B^b_1(0) \subset B^a_{1/C_2}(0)$ since $B^b_{C_2}(0) \subset B^a_1(0)$. To prove this we need to show that if $\|x\|_b < 1$, then $\|x\|_a < 1/C_2$. But if $\|x\|_b < 1$ it follows that

$$\|C_2 x\|_b = C_2\|x\|_b < C_2.$$

Since we assume $B^b_{C_2}(0) \subset B^a_1(0)$, this further implies that $\|C_2 x\|_a < 1$. Dividing both sides by C_2, we obtain

$$\|x\|_a < \frac{1}{C_2},$$

which is what we needed to show.